

NONSYMMETRIC DIFFERENCE WHITTAKER FUNCTIONS

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ABSTRACT. Starting with nonsymmetric global difference spherical functions, we define and calculate spinor (nonsymmetric) global q -Whittaker functions for arbitrary reduced root systems, which are reproducing kernels of the DAHA-Fourier transforms of Nil-DAHA and solutions of the q -Toda-Dunkl eigenvalue problem. We introduce the spinor q -Toda-Dunkl operators as limits of the difference Dunkl operators in DAHA theory under the spinor variant of the Ruijsenaars procedure. Their general algebraic theory (any reduced root systems) is the key part of this paper, based on the new technique of W -spinors and corresponding developments in combinatorics of affine root systems.

Key words: *Root systems; Hecke algebras; Whittaker functions; Toda operators; Macdonald polynomials*

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0. INTRODUCTION

This paper is devoted to the theory of nonsymmetric (spinor) difference Whittaker functions and the corresponding Toda-Dunkl operators in the q -Toda theory for arbitrary irreducible reduced root systems, generalizing the A_1 -case considered in [CM] and [CO1, CO2]. Our approach is based on the new technique of W -spinors, multicomponent functions indexed by the elements of the nonaffine Weyl group W with the natural action of W on the indices.

The applications of this technique are deep (despite its simple definition), with important links to the classical harmonic analysis on symmetric spaces and the theory of spherical, Whittaker and Bessel functions. For instance, spinors arise in the study of nonsymmetric or singular symmetric solutions of symmetric systems such as the Quantum Many-Body Problem; see [C1, Op, CM]. However the main applications so far are for Dunkl-type operators; the Toda-Dunkl operators simply cannot be defined without W -spinors.

The theory of *global* nonsymmetric spherical functions from [C5]—the reproducing kernels of DAHA-Fourier transforms — is the starting point of this paper. They are eigenfunctions of the difference Dunkl operators, but this is just one of their remarkable properties. We continue their general theory (e.g. Proposition 3.2) and then define global nonsymmetric q -Whittaker functions as the limits of global spherical functions using a W -spinor variant of the Ruijsenaars limiting procedure. These functions appear certain quadratic-type generating functions of the q -Hermite polynomials, denoted by \overline{E}_b in the paper, for b from the weight lattice P . See Theorem 3.4 and Proposition 3.3.

One can expect such a generating function to be a series in terms of $q^{(x,b)}\overline{E}_b(\Lambda)$ for generic $x \in \mathbb{C}^n$ and all weights $b \in P$. However, this is not the case. The asymptotic behavior of such series is inconsistent with the analytic Whittaker theory, where the presence of $q^{(x,b)}$ can be expected only for (anti-)dominant b . The “right” generating function requires W -spinors.

The symmetric theory. Symmetric global q -Whittaker functions from [C10] solve the q -Toda eigenvalue problem; the q -Toda operators are due to Ruijsenaars for GL_n [Ru] and Etingof, Sevostyanov [Et, Sev] for any root systems (via Quantum Groups). The corresponding *global* eigenfunctions are given in terms of $\overline{E}_b(\Lambda)$ for anti-dominant $b \in P_-$ only; then $\overline{E}_b(\Lambda)$ become W -invariant. They are *not* W -invariant

(symmetric) in terms of $X = q^x$, just like the classical Whittaker functions. Calling them “symmetric” can be confusing, though they are indeed W -invariant with respect to Λ .

These functions generalize the Whittaker functions from the classical harmonic analysis on symmetric spaces [GW, Wa] and their p -adic counterparts from [CS] given by Shintani-type formulas. W -spinors are not needed in their definition (they are functions, not spinors).

It is necessary to mention a connection with the q -Whittaker functions obtained in [GiL] from the quantum K -theory of the flag varieties; see also [GLO1], [CO1, CO2] and [BeF, BrF]. Establishing the relation to our global “symmetric” q -Whittaker functions is essentially equivalent to the theory of Harish-Chandra decompositions of q, t -spherical and q -Whittaker functions as weighted sums (over W) of their asymptotic expansions; see [HC, C10, Sto, CO1].

It is essential here (and in other geometric applications) that the q -Hermite polynomials coincide with the level-one Demazure characters for affine Kac-Moody algebras (for all weights, not only dominant), due to [San, Ion]. Thus, the present paper solves the algebraic-geometric problem of finding the generating function for *all* level-one Demazure characters; the answer is the spinor q -Whittaker function.

We note that the theory of global q -functions is actually very algebraic (in contrast to the classical differential theory), including special algebraic techniques in the difference Harish-Chandra theory. The Harish-Chandra-type theory of asymptotic decompositions for nonsymmetric (spinor) global spherical and Whittaker functions, including the p -adic limit ($q \rightarrow 0$), will be a subject of our further paper(s). The theory becomes even more algebraic in the nonsymmetric setting due to the use of DAHA intertwining operators, the main tool in the theory of nonsymmetric Macdonald polynomials and its variants/applications.

The main results. The construction of Toda-Dunkl operators is the key result of this paper. The global spinor Whittaker functions are their eigenfunctions, though we can avoid this fundamental connection in the theory of these operators. We define the operator spinor Ruijsenaars-type procedure, which results in a completely algebraic (though involved) theory of the Toda-Dunkl operators.

We give two related approaches to calculating these operators. They are based on Parts A and B of Proposition 5.1 and Lemma 5.3. Using

Part B enables one to obtain arbitrary Toda-Dunkl operators, but not in a very explicit way. Part A provides the formulas for certain basic Toda-Dunkl operators, including those for minuscule weights (which are involved even for A_n — see Section 5.5 for some examples), which are then used to calculate *all* operators; see Proposition 5.4.

The formula for the global spinor Whittaker function for this function and its interpretation via the DAHA-Fourier transform is the second key result of this paper. This provides the most direct approach to justification of the existence of Toda-Dunkl operators, however inconvenient for clarifying their structure. This is in sharp contrast with previously known families of Dunkl operators, where such operators directly resulted from the existence of the polynomial representation of the corresponding Double Affine Hecke Algebras, DAHA.

The construction of Toda-Dunkl operators in [CM] was a surprising development not expected by specialists. The families of Dunkl operators known at that time served only W -invariant families of operators; QMBP, the Quantum Many-Body Problem (also called the Heckman-Opdam system in the differential setting), is a major and the most universal example. In contrast to QMBP, the Toda operators are not symmetric, which makes their theory very different. Only the case of A_1 was considered in [CM] and its continuation [CO1, CO2]; it was not clear after these papers how to generalize the formulas for the one-dimensional spinor Toda-Dunkl operators obtained there.

The general theory of nonsymmetric Whittaker functions follows essentially that for A_1 , though a significant development of the theory of nonsymmetric Hermite polynomials was necessary (as was expected). However, the intrinsic theory of spinor Toda-Dunkl operators (without using the global q -Whittaker functions) required new tools, which deserve thorough analysis and may have applications beyond our paper.

Perspectives. We consider this paper a major step toward the general theory (for arbitrary root systems) in the following directions:

a) *The theory of pseudo-polynomial representations of Nil-DAHA*, which will explain the algebraic origins of the Toda-Dunkl operators. They are *not* induced representations of nil-DAHA in contrast to the polynomial DAHA modules in all other theories. However, as in [CO2] (the case of A_1), they are induced modules of the *core subalgebras* of Nil-DAHA; the Toda-Dunkl operators naturally act there. This construction is part of *the theory of canonical-crystal bases of Nil-DAHA*

(in process now), which is expected to have significant applications in the representation theory of DAHA and beyond. The theory of canonical-crystal bases was a major development in the theory of *quantum groups*, closely related to *Kazhdan-Lusztig polynomials* (missing in the DAHA theory so far) and *cluster algebras*.

b) *The analytic theory of global nonsymmetric q, t -spherical and q -Whittaker functions*, including the Harish-Chandra theory of their decompositions in terms of the asymptotic expansions. This theory requires the development of analytic techniques for dealing with W -spinors. The first application (already reached) is a nonsymmetric generalization of the existence of *the asymptotic expansions of global spherical functions* from [Sto] (see [CO1] for the case of A_1), which includes the global q -Whittaker functions as well. The nonsymmetric methods significantly simplify considerations here, similar to their role at all other levels of the theory of DAHA and Macdonald polynomials.

c) *Applications of “symmetric” q -Whittaker functions*, including the K -theory of flag varieties [GiL], the theory of affine flag varieties [BrF], the theory of Demazure characters and local and global Weyl modules in the Kac-Moody theory, Rogers-Ramanujan identities (see [CF] and the references there), q -Whittaker processes (random discrete polymers, see [BC]) and, presumably, the quantum Langlands program. We expect these directions to be enriched by our theory of spinor q -Whittaker functions and Toda-Dunkl operators. One of the applications (last but not least), which was outlined in [CM] for A_1 , will be the theory of “nonsymmetric” *Matsumoto-type p -adic Whittaker functions* in the classical p -adic harmonic analysis.

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1. DOUBLE HECKE ALGEBRA

Let $R = \{\alpha\} \subset \mathbb{R}^n$ be a root system of type A, B, \dots, F, G with respect to a Euclidean form (z, z') on $\mathbb{R}^n \ni z, z'$, W the Weyl group generated by the reflections s_α , R_+ the set of positive roots ($R_- = -R_+$)

corresponding to fixed simple roots $\alpha_1, \dots, \alpha_n$, Γ the Dynkin diagram with $\{\alpha_i, 1 \leq i \leq n\}$ as the vertices. Accordingly,

$$R^\vee = \{\alpha^\vee = 2\alpha/(\alpha, \alpha)\}.$$

The root lattice and the weight lattice are:

$$Q = \oplus_{i=1}^n \mathbb{Z}\alpha_i \subset P = \oplus_{i=1}^n \mathbb{Z}\omega_i,$$

where $\{\omega_i\}$ are fundamental weights: $(\omega_i, \alpha_j^\vee) = \delta_{ij}$ for the simple coroots α_i^\vee . Replacing \mathbb{Z} by $\mathbb{Z}_\pm = \{m \in \mathbb{Z}, \pm m \geq 0\}$ we obtain Q_\pm, P_\pm . Here and further, see [B].

The form will be normalized by the condition $(\alpha, \alpha) = 2$ for the *short* roots in this paper. Thus,

$$\nu_R = \{\nu_\alpha \stackrel{\text{def}}{=} (\alpha, \alpha)/2, \alpha \in R\} \text{ can be either } \{1\}, \{1, 2\}, \text{ or } \{1, 3\}.$$

We will use the notation ν_{lng} for long roots and $\nu_{\text{sht}} = 1$ for short roots.

The normalization leads to the inclusions $Q \subset Q^\vee, P \subset P^\vee$, where P^\vee is generated by the fundamental coweights $\{\omega_i^\vee\}$ dual to $\{\alpha_i\}$.

We set $\nu_i = \nu_{\alpha_i}$ and

$$(1.1) \quad \rho_\nu \stackrel{\text{def}}{=} \frac{1}{2} \sum_{\nu_\alpha = \nu} \alpha = \sum_{\nu_i = \nu} \omega_i, \text{ where } \alpha \in R_+, \nu \in \nu_R.$$

Accordingly, $(\rho_\nu, \alpha_i^\vee) = 1$ for $\nu_i = \nu$. Together with ρ_ν , we will also use the notation ρ_\diamond for $\diamond = \text{lng}, \text{sht}$.

1.1. Affine Weyl group. The vectors $\tilde{\alpha} = [\alpha, \nu_\alpha j] \in \mathbb{R}^n \times \mathbb{R} \subset \mathbb{R}^{n+1}$ for $\alpha \in R, j \in \mathbb{Z}$ form the *affine root system* $\tilde{R} \supset R$ ($z \in \mathbb{R}^n$ are identified with $[z, 0]$). We add $\alpha_0 \stackrel{\text{def}}{=} [-\vartheta, 1]$ to the simple roots for the *maximal short root* $\vartheta \in R_+$. It is also the *maximal positive coroot* because of the choice of normalization.

The corresponding set \tilde{R}_+ of positive roots equals $R_+ \cup \{[\alpha, \nu_\alpha j], \alpha \in R, j > 0\}$. Indeed, any positive affine root $[\alpha, \nu_\alpha j]$ is a linear combination of $\{\alpha_i, 0 \leq i \leq n\}$ with coefficients from \mathbb{Z}_+ .

We complete the Dynkin diagram Γ of R by adding α_0 ($-\vartheta$, to be more exact); it is called *affine Dynkin diagram* $\tilde{\Gamma}$. One can obtain it from the completed Dynkin diagram from [B] for the *dual system* R^\vee by reversing all arrows. The number of laces between α_i and α_j in $\tilde{\Gamma}$ will be denoted by m_{ij} .

The set of indices of the images of α_0 by all the automorphisms of $\tilde{\Gamma}$ will be denoted by O ($O = \{0\}$ for E_8, F_4, G_2). Let $O' = \{r \in O, r \neq$

$0\}$. The elements ω_r for $r \in O'$ are the minuscule weights: $(\omega_r, \alpha^\vee) \leq 1$ for all $\alpha \in R_+$.

Given $\tilde{\alpha} = [\alpha, \nu_\alpha j] \in \tilde{R}$ and $b \in P$, let

$$(1.2) \quad s_{\tilde{\alpha}}(\tilde{z}) = \tilde{z} - (z, \alpha^\vee)\tilde{\alpha}, \quad b'(\tilde{z}) = [z, \zeta - (z, b)]$$

for $\tilde{z} = [z, \zeta] \in \mathbb{R}^{n+1}$.

The *affine Weyl group* \tilde{W} is generated by all $s_{\tilde{\alpha}}$ for $\tilde{\alpha} \in \tilde{R}_+$; we write $\tilde{W} = \langle s_{\tilde{\alpha}}(\tilde{\alpha} \in \tilde{R}_+) \rangle$. One can take the simple reflections $s_i = s_{\alpha_i}$ ($0 \leq i \leq n$) as its generators and introduce the corresponding notion of the length. The group \tilde{W} is the semidirect product $W \ltimes Q'$ of its subgroups $W = \langle s_\alpha, \alpha \in R_+ \rangle$ and $Q' = \{a', a \in Q\}$, where

$$(1.3) \quad \alpha' = s_\alpha s_{[\alpha, \nu_\alpha]} = s_{[-\alpha, \nu_\alpha]} s_\alpha \quad \text{for } \alpha \in R.$$

The *extended affine Weyl group* \widehat{W} generated by W and P' (instead of Q') is isomorphic to $W \ltimes P'$:

$$(1.4) \quad (wb')([z, \zeta]) = [w(z), \zeta - (z, b)] \quad \text{for } w \in W, b \in B.$$

From now on, b and b' , P and P' will be identified.

Given $b \in P_+$, let w_0^b be the longest element in the subgroup $W^b \subset W$ of the elements preserving b . This subgroup is generated by simple reflections. We set

$$(1.5) \quad u_b = w_0 w_0^b \in W, \quad \pi_b = b(u_b)^{-1} \in \widehat{W}, \quad u_i = u_{\omega_i}, \quad \pi_i = \pi_{\omega_i},$$

where w_0 is the longest element in W , $1 \leq i \leq n$.

The elements $\pi_r \stackrel{\text{def}}{=} \pi_{\omega_r}$ ($r \in O'$) and $\pi_0 = \text{id}$ leave $\tilde{\Gamma}$ invariant and form a group denoted by Π , which is isomorphic to P/Q by the natural projection $\{\omega_r \mapsto \pi_r\}$. As to u_r ($r \in O'$), they preserve the set $\{-\vartheta\} \cup \{\alpha_i, i > 0\}$. The relations $\pi_r(\alpha_0) = \alpha_r = (u_r)^{-1}(-\vartheta)$ distinguish the indices $r \in O'$. Moreover, one has

$$(1.6) \quad \widehat{W} = \Pi \ltimes \tilde{W}, \quad \text{where } \pi_r s_i \pi_r^{-1} = s_j \text{ if } \pi_r(\alpha_i) = \alpha_j, \quad i, j \geq 0.$$

We note that $\pi_r^{-1} = \pi_{r^*}$ and $u_r^{-1} = u_{r^*}$ for $r \in O'$, where $r^* \in O'$ is determined by the action of $-w_0$ on the nonaffine Dynkin diagram Γ : $-w_0(\alpha_r) = \alpha_{r^*}$.

We will need the following *affine action* of \widehat{W} on $z \in \mathbb{R}^n$:

$$(1.7) \quad \begin{aligned} (wb)((z)) &= w(b+z), \quad w \in W, \quad b \in P, \\ s_{\tilde{\alpha}}((z)) &= z - ((z, \alpha^\vee) + j)\alpha, \quad \tilde{\alpha} = [\alpha, \nu_\alpha j] \in \tilde{R}. \end{aligned}$$

For instance, $(bw)((0)) = b$ for any $w \in W$. The relation to the above action is given in terms of the *affine pairing* $([z, l], z' + d) \stackrel{\text{def}}{=} (z, z') + l :$

$$(1.8) \quad (\widehat{w}([z, l]), \widehat{w}([z', l]) + d) = ([z, l], z' + d) \text{ for } \widehat{w} \in \widehat{W},$$

where we treat d formally.

1.2. The length on \widehat{W} . Setting $\widehat{w} = \pi_r \widetilde{w} \in \widehat{W}$ ($\pi_r \in \Pi, \widetilde{w} \in \widetilde{W}$), the length $l(\widehat{w})$ is by definition the length of the reduced decomposition $\widetilde{w} = s_{i_l} \cdots s_{i_2} s_{i_1}$ in terms of the simple reflections $s_i, 0 \leq i \leq n$. The number of s_i in this decomposition such that $\nu_i = \nu$ is denoted by $l_\nu(\widehat{w})$. We will also use the notation $l_\diamond(\widehat{w})$ for $\diamond = \text{lng, sht}$.

The *length* can be also defined as the cardinality $|\lambda(\widehat{w})|$ of the λ -set of \widehat{w} :

$$(1.9) \quad \lambda(\widehat{w}) \stackrel{\text{def}}{=} \widetilde{R}_+ \cap \widehat{w}^{-1}(\widetilde{R}_-) = \{\tilde{\alpha} \in \widetilde{R}_+, \widehat{w}(\tilde{\alpha}) \in \widetilde{R}_-\}, \widehat{w} \in \widehat{W}.$$

One has:

$$(1.10) \quad \lambda(\widehat{w}) = \cup_\nu \lambda_\nu(\widehat{w}), \lambda_\nu(\widehat{w}) \stackrel{\text{def}}{=} \{\tilde{\alpha} \in \lambda(\widehat{w}), \nu(\tilde{\alpha}) = \nu\}.$$

The coincidence with the previous definition is based on the equivalence of the *length equality*

$$(1.11) \quad (a) \quad l_\nu(\widehat{w}\widehat{u}) = l_\nu(\widehat{w}) + l_\nu(\widehat{u}) \text{ for } \widehat{w}, \widehat{u} \in \widehat{W}$$

and the *cocycle relation*

$$(1.12) \quad (b) \quad \lambda_\nu(\widehat{w}\widehat{u}) = \lambda_\nu(\widehat{u}) \cup \widehat{u}^{-1}(\lambda_\nu(\widehat{w})),$$

which, in turn, is equivalent to the *positivity condition*

$$(1.13) \quad (c) \quad \widehat{u}^{-1}(\lambda_\nu(\widehat{w})) \subset \widetilde{R}_+$$

and is also equivalent to the *embedding condition*

$$(1.14) \quad (d) \quad \lambda_\nu(\widehat{u}) \subset \lambda_\nu(\widehat{w}).$$

See, e.g., [C4, C8] and also [B, Hu]. Applying (1.12) to the reduced decomposition $\widehat{w} = \pi_r s_{i_l} \cdots s_{i_2} s_{i_1}$, we obtain an ordering of the λ -set:

$$(1.15) \quad \lambda(\widehat{w}) = \{ \tilde{\alpha}^1 = \alpha_{i_1}, \tilde{\alpha}^2 = s_{i_1}(\alpha_{i_2}), \tilde{\alpha}^3 = s_{i_1} s_{i_2}(\alpha_{i_3}), \dots, \tilde{\alpha}^l = \widetilde{w}^{-1} s_{i_l}(\alpha_{i_l}) \}.$$

We will call (1.15) the λ -sequence associated with the given decomposition of \widehat{w} . Such sequences are exactly those in \widetilde{R}_+ satisfying properties

(i, ii) from the following lemma. We consider $\lambda(\widehat{w})$ as sets (not sequences) in quite a few statements below, which then depend only on elements \widehat{w} (not on their reduced decompositions).

Lemma 1.1. *Given a reduced decomposition $\widehat{w} = \pi_r s_{j_l} \cdots s_{j_1} \in \widehat{W}$, form the λ -sequence $\lambda(\widehat{w}) = \{\tilde{\alpha}^1, \dots, \tilde{\alpha}^l\}$ using (1.15).*

(i) *If $\tilde{\alpha} = \tilde{\alpha}^q + \tilde{\alpha}^r \in \tilde{R}_+$, then $\tilde{\alpha} = \tilde{\alpha}^p$ for some p between q and r . The same holds if $\tilde{\alpha} = c_1 \tilde{\alpha}^q + c_2 \tilde{\alpha}^r \in \tilde{R}_+$ for positive rational c_1, c_2 .*

(ii) *If $\lambda(\widehat{w}) \ni \tilde{\alpha} = \tilde{\beta} + \tilde{\gamma}$ for $\tilde{\beta}, \tilde{\gamma} \in \tilde{R}_+ \cup [0, \mathbb{Z}_+]$, then at least one of $\tilde{\beta}, \tilde{\gamma}$ belongs to $\lambda(\widehat{w})$ and exactly one of $\tilde{\beta}, \tilde{\gamma}$ comes before $\tilde{\alpha}$ in $\lambda(\widehat{w})$.*

See Main Theorem 2.1 of [C9]. Note that the corresponding reduced decomposition of \widehat{w} can be uniquely recovered from the λ -sequence $\lambda(\widehat{w})$; considered as a λ -set, the latter is sufficient to recover \widehat{w} .

Reduction modulo W . The following proposition generalizes the construction of the elements π_b for $b \in P_+$; see [C4] or [C8].

Proposition 1.2. *Given $b \in P$, there exists a unique decomposition $b = \pi_b u_b$, $u_b \in W$ satisfying one of the following equivalent conditions:*

- (i) $l(\pi_b) + l(u_b) = l(b)$ and $l(u_b)$ is the greatest possible,
- (ii) $\lambda(\pi_b) \cap R = \emptyset$.

The latter condition implies that $l(\pi_b) + l(w) = l(\pi_b w)$ for any $w \in W$. Besides, the relation $u_b(b) \stackrel{\text{def}}{=} b_- \in P_- = -P_+$ holds, which, in turn, determines u_b uniquely if one of the following equivalent conditions is imposed:

- (iii) $l(u_b)$ is the smallest possible,
- (iv) if $\alpha \in \lambda(u_b)$ then $(\alpha, b) \neq 0$.

□

Condition (ii) readily gives a complete description of the set $\pi_P = \{\pi_b, b \in P\}$, namely, only $[\alpha < 0, \nu_\alpha j > 0]$ can appear in $\lambda(\pi_b)$.

Explicitly,

$$(1.16) \quad \lambda(b) = \{\tilde{\alpha} > 0, (b, \alpha^\vee) > j \geq 0 \text{ if } \alpha \in R_+, \\ (b, \alpha^\vee) \geq j > 0 \text{ if } \alpha \in R_-\},$$

$$(1.17) \quad \lambda(\pi_b) = \{\tilde{\alpha} > 0, \alpha \in R_-, (b_-, \alpha^\vee) > j > 0 \text{ if } u_b^{-1}(\alpha) \in R_+, \\ (b_-, \alpha^\vee) \geq j > 0 \text{ if } u_b^{-1}(\alpha) \in R_-\},$$

For instance, $l(b) = l(b_-) = -2(\rho^\vee, b_-)$ for $2\rho^\vee = \sum_{\alpha > 0} \alpha^\vee$.

Switching here to l_\diamond for $\diamond = \text{lng}, \text{sht}$, one has $l_\diamond(b) = -2(\rho_\diamond^\vee, b_-)$, where

$$\rho_\diamond^\vee = \frac{1}{2} \sum_{\alpha > 0, \nu_\alpha = \nu_\diamond} \alpha^\vee = \rho_\diamond / \nu_\diamond.$$

The element $b_- = u_b(b)$ is the unique element from P_- that belongs to the orbit $W(b)$. Thus the equality $c_- = b_-$ means that b, c belong to the same orbit. We will also use $b_+ \stackrel{\text{def}}{=} w_0(b_-)$, the unique element in $W(b) \cap P_+$. In terms of π_b ,

$$u_b \pi_b = b_-, \quad \pi_b u_b = b_+.$$

Note that $l_\diamond(\pi_b w) = l_\diamond(\pi_b) + l_\diamond(w)$ for all $b \in P$, $w \in W$. For instance,

$$(1.18) \quad \begin{aligned} l_\diamond(b_- w) &= l_\diamond(b_-) + l_\diamond(w), \quad l_\diamond(w b_+) = l_\diamond(b_+) + l_\diamond(w), \\ l_\diamond(u_b \pi_b w) &= l_\diamond(u_b) + l_\diamond(\pi_b) + l_\diamond(w) \quad \text{for } b \in P, w \in W. \end{aligned}$$

Partial orderings on P . The following two partial orderings on P are commonly used in the theory of Dunkl operators and nonsymmetric Macdonald polynomials. See [C3, Op, M2].

We mainly need the partial ordering on P defined by:

$$(1.19) \quad \begin{aligned} b \preceq c, c \succeq b \quad &\text{if } b_- < c_- \text{ or } \{b_- = c_- \text{ and } b \leq c\}, \\ \text{where } b \leq c, c \geq b \quad &\text{for } b, c \in P \text{ if } c - b \in Q_+. \end{aligned}$$

Recall that $b_- = c_-$ means that b, c belong to the same W -orbit. We write $<, >, \prec, \succ$ respectively if $b \neq c$. This ordering was also used in [C3] in the process of calculating the coefficients of the Y -operators.

For any $b \in P$, we define the sets

$$(1.20) \quad \begin{aligned} \sigma(b) &\stackrel{\text{def}}{=} \{c \in P, c \succeq b\}, \quad \sigma_*(b) \stackrel{\text{def}}{=} \{c \in P, c \succ b\}, \\ \sigma_-(b) &\stackrel{\text{def}}{=} \sigma(b_-), \quad \sigma_+(b) \stackrel{\text{def}}{=} \sigma_*(b_+) = \{c \in P, c_- > b_-\}. \end{aligned}$$

The second partial ordering is defined by

$$(1.21) \quad b \preceqslant c, c \succeqslant b \quad \text{if } b_- < c_- \text{ or } \{b_- = c_- \text{ and } u_b \leq u_c\},$$

where u_b is from Proposition 1.2 and \leq applied to elements of W is the Bruhat ordering.

It is not hard to show that $c \succeqslant b \Rightarrow c \succeq b$ and that the converse is false (see [M3, (2.7.7)]). We remark that, if w_b is the unique shortest

element of W satisfying $w_b(b_+) = b$, then $u_b \leq u_c$ if and only if $w_c \leq w_b$.

1.3. On λ -sequences of reflections. The construction of the Toda-Dunkl operators will heavily use the sequences $\lambda(s_{\tilde{\alpha}})$ for reflections $s_{\tilde{\alpha}}$, where $\tilde{\alpha} = [\alpha, \nu_\alpha j] \in \tilde{R}_+$. As above, λ -sequences will be frequently considered as sets, i.e. without the orderings determined by reduced decompositions; these sequences are described intrinsically by Lemma 1.1. Formula (1.19) from [C9] states that all λ -sequences for $s_{\tilde{\alpha}}$ are as follows:

$$(1.22) \quad \lambda(s_{\tilde{\alpha}}) = \{s_{\tilde{\alpha}}(-\lambda(\tilde{w}))\}_{op} \cup \tilde{\alpha} \cup \lambda(\tilde{w}) \text{ (as sequences),}$$

where $\tilde{w} \in \tilde{W}$ is of minimal possible length such that $\tilde{w}(\tilde{\alpha}) = \alpha_m$ among all $m \geq 0$; by $\{\cdot\}_{op}$, we mean the inversion of the ordering of a given sequence $\{\cdot\}$. Then $s_{\tilde{\alpha}} = \tilde{w}^{-1}s_m\tilde{w}$ is reduced for any reduced decomposition of \tilde{w} and an arbitrary reduced decomposition of $s_{\tilde{\alpha}}$ can be presented in this form for proper \tilde{w} , s_m satisfying the above minimality condition:

$$(1.23) \quad s_{\tilde{\alpha}} = s_{j_1} \cdots s_{j_p} s_m s_{j_p} \cdots s_{j_1}, \text{ where}$$

$$(1.24) \quad \tilde{w} = s_{j_p} \cdots s_{j_1}, p = l(\tilde{w}), j_1, \dots, j_p \geq 0.$$

See e.g. Proposition 1.1 from [C9] for a proof of these (standard) facts. We observe that if such a reduced decomposition is used to construct the λ -sequence $\lambda(s_{\tilde{\alpha}}) = \{\tilde{\beta}^1, \dots, \tilde{\beta}^l\}$, where $l = l(s_{\tilde{\alpha}}) = 2p + 1$, then

$$(1.25) \quad \tilde{\beta}^{l-i+1} = -s_{\tilde{\alpha}}(\tilde{\beta}^i), \text{ for } 1 \leq i \leq l.$$

Furthermore, for any $\hat{w} \in \widehat{W}$ and any sequence $\lambda(\hat{w})$, one has

$$\tilde{\alpha} \in \lambda(\hat{w}) \Leftrightarrow \lambda(s_{\tilde{\alpha}}) \setminus \{\tilde{\alpha}\} = \bigcup_{\tilde{\beta}} \{\tilde{\beta}, \tilde{\beta}' = -s_{\tilde{\alpha}}(\tilde{\beta})\} \text{ (as sets), where}$$

$$(1.26) \quad \tilde{\beta} \in \lambda(\hat{w}) \cap \lambda(s_{\tilde{\alpha}}) \text{ such that } \tilde{\beta} \text{ appears in } \lambda(\hat{w}) \text{ before } \tilde{\alpha}.$$

See formula (1.20) in [C9]. Here $\tilde{\beta}$ and $\tilde{\beta}' = -s_{\tilde{\alpha}}(\tilde{\beta})$ do not coincide unless $\tilde{\beta} = \tilde{\alpha}$. Indeed, $\tilde{\beta}' = 2\frac{(\alpha, \tilde{\beta})}{(\alpha, \alpha)}\tilde{\alpha} - \tilde{\beta}$ and $\tilde{\beta}' = \tilde{\beta}$ if and only if $\tilde{\beta}$ is proportional to $\tilde{\alpha}$, which occurs exactly for $\tilde{\beta} = \tilde{\alpha}$. All pairs $\{\tilde{\beta}, \tilde{\beta}'\}$ are pairwise distinct. Indeed, if $\tilde{\gamma} = \tilde{\beta}'$ for any $\tilde{\beta}, \tilde{\gamma} \in \lambda(s_{\tilde{\alpha}})$, then $\tilde{\alpha}$, which is $c(\tilde{\beta} + \tilde{\beta}')$ for $c > 0$, occurs between $\tilde{\beta}$ and $\tilde{\gamma}$ in this sequence, which is impossible by construction. Here and below, see Lemma 1.1.

Let us list some other properties of the sets $\lambda(s_{\tilde{\alpha}})$. First of all, always $(\alpha, \beta) > 0$ for any $\tilde{\beta} = [\beta, \nu_{\beta}k] \in \lambda(s_{\tilde{\alpha}})$ and we have the following equivalent inequalities :

$$(1.27) \quad 2 \frac{(\alpha, \beta)}{(\alpha, \alpha)} \nu_{\alpha} j \geq \nu_{\beta} k \iff (\alpha, \beta) j \geq \nu_{\beta} k \iff 2 \frac{(\alpha, \beta)}{(\beta, \beta)} j \geq k.$$

The strict inequality $(\alpha, \beta) j > \nu_{\beta} k$ for $\tilde{\beta} > 0$ gives that $\tilde{\beta} \in \lambda(s_{\tilde{\alpha}})$. If $(\alpha, \beta) j = \nu_{\beta} k$, then the conditions $(\alpha, \beta) > 0$ and $s_{\alpha}(\beta) < 0$ are necessary and sufficient.

If $\beta \neq \alpha$, then the inequality in (1.27) becomes $j \geq k$ unless β is short and α is long; in the latter case, it becomes $\nu_{\alpha} j \geq k$. If $\beta = \alpha$, then $\tilde{\beta} = [\alpha, \nu_{\alpha}k]$ are in $\lambda(s_{\tilde{\alpha}})$ if and only if $0 \leq k \leq 2j$ when $\alpha > 0$, and $0 < k < 2j$ when $\alpha < 0$.

Following the calculation of the sets $\lambda(s_{\tilde{\alpha}})$ for $\alpha \in \tilde{R}_-, j > 0$ in formula (1.28) from [C9] and the action of $-s_{\tilde{\alpha}}$ in these sets described there, let us calculate such sets for arbitrary positive affine roots.

Lemma 1.3. *For $\tilde{\alpha} = [\alpha, \nu_{\alpha}j] \in \tilde{R}_+$ and $\diamond = \text{sht}, \text{lng}$, we set*

$$\delta_{\alpha, \diamond} = \delta_{\nu_{\alpha}, \nu_{\diamond}}, \quad \eta_{\alpha \diamond} = 1 \text{ unless } \eta_{\alpha \diamond} = \nu_{\text{lng}} \text{ when } \nu_{\alpha} = 1 = \nu_{\text{sht}} \text{ and } \diamond = \text{lng}.$$

(i) *For $\alpha \in R_+$, there exists a set $\{\beta^i\} \subset R_+ \setminus \{\alpha\}$ such that*

$$(1.28) \quad \lambda_{\diamond}(s_{\alpha}) \setminus \{\alpha\} = \{ \{\beta^i, -s_{\alpha}(\beta^i)\} \mid \nu_{\beta^i} = \nu_{\diamond}, s_{\alpha}(\beta^i) < 0 \},$$

$$\text{where } 1 \leq i \leq \frac{l_{\diamond}(s_{\alpha}) - \delta_{\alpha, \diamond}}{2}, \quad l_{\diamond}(s_{\alpha}) = 2 \frac{(\alpha, \rho_{\diamond})}{\eta_{\alpha \diamond} \nu_{\alpha}} - \delta_{\alpha, \diamond}.$$

More explicitly, $l_{\diamond}(s_{\alpha}) = 2(\alpha^{\vee}, \rho_{\diamond}) - \delta_{\alpha, \diamond}$ for long roots $\alpha \in R_+$ and $l_{\diamond}(s_{\alpha}) = 2(\alpha, \rho_{\diamond}^{\vee}) - \delta_{\alpha, \diamond}$ for short α .

(ii) *Let $\tilde{\alpha} = [\alpha, \nu_{\alpha}j]$ for $\alpha \in R_+, j > 0$. Then provided that $\tilde{\beta} = [\beta, \nu_{\beta}k] > 0$ such that $\beta \neq \alpha$, $\nu_{\beta} = \nu_{\diamond}$ and $(\alpha, \beta) > 0$,*

$$(1.29) \quad \lambda_{\diamond}(s_{\alpha}) = \{ [\alpha, \nu_{\alpha}k], 0 \leq k \leq 2j \} \cup \{ \tilde{\beta} > 0, 0 \leq \nu_{\beta}k < (\alpha, \beta)j \} \\ \cup \{ [\beta, (\alpha, \beta)j]; \beta < 0 \text{ or } \beta \in \lambda(s_{\alpha}) \}.$$

Accordingly, $l_{\diamond}(s_{\tilde{\alpha}}) = l_{\diamond}(j\alpha) + l_{\diamond}(s_{\alpha})$, where $j\alpha = s_{\alpha}s_{\tilde{\alpha}} = s_{[-\alpha, \nu_{\alpha}j]}s_{\alpha}$.

(iii) *Let $\tilde{\alpha} = [-\alpha, \nu_{\alpha}j]$ for $\alpha \in R_+, j > 0$. Then assuming that $\tilde{\beta} = [-\beta, \nu_{\beta}k] > 0$ and that $\nu_{\beta} = \nu_{\diamond}$,*

$$(1.30) \quad \lambda_{\diamond}(s_{\tilde{\alpha}}) = \{ [-\beta, \nu_{\alpha}k] \in \tilde{R}_+, 0 \leq \nu_{\beta}k < (\alpha, \beta)j \} \\ \cup \{ [-\beta, (\alpha, \beta)j]; \beta > 0 < s_{\alpha}(\beta), (\alpha, \beta) > 0 \}.$$

One has $\lambda_\diamond(s_{\tilde{\alpha}}) = \lambda_\diamond(-j\alpha) \setminus \{[-\beta, (\alpha, \beta)j] \mid \beta \in \lambda_\diamond(s_\alpha)\}$.

Proof. The presentation of $\lambda(s_\alpha)$ from (i) is a particular case of (1.26). Then we will use that $\rho_\diamond - w(\rho_\diamond) = \sum_{\beta \in \lambda_\diamond(w)} \beta$, combining it with (1.28) and the formula $\beta' = -s_\alpha(\beta) = \eta_{\alpha\beta}\alpha - \beta$ for $\beta \in \lambda(s_\alpha) \setminus \{\alpha\}$, where $\eta_{\alpha\beta} = 1$ unless $\eta_{\alpha\beta} = \nu_{\text{ing}}$ for short α and long β . One has

$$(1.31) \quad (\rho_\diamond - s_\alpha(\rho_\diamond), \alpha)/\nu_\alpha = 2(\rho_\diamond, \alpha)/\nu_\alpha \\ = \delta_{\alpha, \diamond} + \sum_{\beta \in \lambda_\diamond(s_\alpha)} \left(\frac{\eta_{\alpha\beta}\alpha}{2}, \alpha\right)/\nu_\alpha = \delta_{\alpha, \diamond} + \sum_{\beta \in \lambda_\diamond(s_\alpha)} \eta_{\alpha\beta} = \delta_{\alpha, \diamond} + \eta_{\alpha\alpha} l_\diamond(s_\alpha).$$

Claim (ii) follows from the previous considerations and formula (1.16). One can also use that $\lambda(s_\alpha) \cap \lambda(-j\alpha) = \emptyset$. Claim (iii) is formula (1.28) from [C9]. It follows from (i) or (ii) using the decomposition $s_{[-\alpha, \nu_\alpha j]} = s_\alpha \cdot (-j\alpha)$; here $-j\alpha = s_{[-\alpha, \nu_\alpha j]} s_\alpha$ is reduced. \square

Nonaffine reflections. Let $s_\alpha = w^{-1}s_i w = s_{j_1} \cdots s_{j_p} s_m s_{j_p} \cdots s_{j_1}$ be a reduced decomposition from (1.23) for $\alpha \in R_+$ and proper $m > 0$; here $w = s_{j_p} \cdots s_{j_1}$, $l(w) = p$. Then one has the inequalities

$$(1.32) \quad (\alpha, \beta) > 0 \text{ for } \beta \in \lambda(w), \text{ equivalently,} \\ (\alpha_m, \beta') < 0 \text{ for } \beta' = -w(\beta) \in \lambda(w^{-1}) = -w(\lambda(w)).$$

Lemma 1.4. *For any $\alpha \in R_+$, let $w = s_{j_k} \cdots s_{j_1}$ be a reduced decomposition of an element $w \in W$ such that $w(\alpha) = \alpha_m$ and $k = l(w)$ is minimal possible among all α_m (which is then $k = (l(s_\alpha) - 1)/2$). Equivalently, $s_\alpha = s_{j_1} \cdots s_{j_k} s_m s_{j_k} \cdots s_{j_1}$ is reduced.*

(i) *For such w , α_m and any reduced decomposition of w , there exists its extension $\tilde{w} = s_{j_p} \cdots s_{j_1}$ of length $p = (l(s_{\theta'}) - 1)/2$ and the corresponding reduced decomposition $s_{\theta'} = s_{j_1} \cdots s_{j_p} s_m s_{j_p} \cdots s_{j_1}$, where θ' is the maximal root θ for long α and ϑ for short α ; $m, j_1, \dots, j_p > 0$.*

(ii) *When $\alpha = \theta'$, any $m = 1, \dots, n$ can be taken here provided that $|\alpha_m| = |\theta'|$; the corresponding element $w = w^{(m)} = s_{j_p} \cdots s_{j_1}$ (but not its reduced decomposition) is uniquely determined by the choice of m , equivalently, by the condition $w(\theta') = \alpha_m$ together with the inequalities $(\beta, \theta') > 0$ for all $\beta \in \lambda(w)$.*

Proof. Let us demonstrate that the inequalities $(\alpha_m, \beta') < 0$ from (1.32) for all $\beta' \in \lambda(w^{-1})$ are actually sufficient to ensure that $l(w)$ is minimal possible among all w such that $w(\alpha) = \alpha_m$ for a given α_m .

We argue by “descending” induction on $l(s_\alpha)$. Unless α satisfying inequalities (1.32) is maximal long or short root, we can find a simple

root α_j such that $(\alpha, \alpha_j) < 0$. Then $\alpha_j \notin \lambda(s_\alpha)$. Therefore $\alpha_j \notin \lambda(w)$, ws_j is reduced, $(\alpha_m, w(\alpha_j)) = (w^{-1}(\alpha_m), \alpha_j) < 0$ and, finally, $\check{w} \stackrel{\text{def}}{=} ws_j$ is of length $l(w) + 1$ satisfying the inequalities from (1.32) for any $\check{\beta}' \in \lambda(\check{w}^{-1}) = w(\alpha_j) \cup \lambda(w^{-1})$. Continuing this way by induction, we eventually construct w satisfying $w(\theta') = \alpha_m$ and the inequalities from (1.32), where $|\theta'| = |\alpha_m|$.

Let $\theta' = \vartheta$ here for the sake of definiteness. We claim that the resulting w is a minimal element satisfying $w(\vartheta) = \alpha_m$; moreover, it is unique (depends only on α_m). Indeed, the subgroup $W^\vartheta = \{u \in W \mid u(\vartheta) = \vartheta\}$ is parabolic generated by simple s_r such that $r \neq k$ for α_k connected with α_0 in the affine Dynkin diagram $\tilde{\Gamma}$ for the (twisted) root system \tilde{R} . We use here that $\vartheta = \omega_k$ unless for A_n , where $\vartheta = \omega_1 + \omega_n$. Due to the inequalities from (1.32), all products wu are reduced for such w and any elements $u \in W^\vartheta$, so w is really minimal and unique such.

Here one can begin with any short simple $\alpha = \alpha_m$, which proves (ii). Moreover, the induction process above automatically guarantees that $\check{w} = ws_j$ has to be minimal (though maybe not unique such) for $\check{\alpha} = s_j(\alpha) = \check{w}^{-1}(\alpha_j)$, as well as for all consecutive w serving $\{\alpha, \check{\alpha}, \dots, \vartheta\}$, since the last w in this chain has been proven to be minimal. This justifies that the inequalities in (1.32) are sufficient for the minimality of w and gives (ii). \square

As a by-product, we obtain that for any reduced decomposition $w = s_{j_p} \cdots s_{j_1}$ of minimal w such that $w(\vartheta) = \alpha_m$, where α_m is any given short root,

$$(1.33) \quad (s_{j_{k+1}} \cdots s_{j_p}(\alpha_m), \alpha_{j_k}^\vee) = (s_{j_k} \cdots s_{j_1}(\vartheta), \alpha_{j_k}^\vee) = -1, \quad 1 \leq k \leq p.$$

Indeed, using (1.32)

$$(s_{j_{k+1}} \cdots s_{j_p}(\alpha_m), \alpha_{j_k}) = (\alpha_m, s_{j_p} \cdots s_{j_{k+1}}(\alpha_{j_k})),$$

where $s_{j_p} \cdots s_{j_{k+1}}(\alpha_{j_k}) \in \lambda(w^{-1})$,

which gives that the right-hand side in (1.33) is negative; so it must be -1 since α_m is short.

It is of interest to calculate explicitly the elements $w^{(m)}$ from (ii) and their λ -sets. Let us do the latter for $m = k$ for short α_k connected with α_0 in $\tilde{\Gamma}$. Then $w'(\vartheta) = \alpha_k$ for $w' = s_\vartheta s_k$ and $\lambda(w') = s_k(\lambda(s_\vartheta) \setminus \alpha_k)$. Dividing w' by the maximal possible $u \in W^\vartheta$ on the right, we obtain

that

$$\lambda(w^{(k)}) = \{ \beta \in R_+ \mid (\beta, \vartheta) > 0, (\beta, \alpha_k) = 0 \}.$$

Indeed, we need to remove $s_k(\beta)$ from $\lambda(w')$ for $\beta \in \lambda(s_\vartheta) \setminus \alpha_k$ such that $(s_k(\beta), \vartheta) = 0 = (\beta, \vartheta - \alpha_k)$. Recall that $\beta + \beta' = \nu_\beta \vartheta$ for $\beta' = -s_\vartheta(\beta)$. Therefore, either $(\beta, \alpha_k) = \nu_\beta$ and $(\beta', \alpha_k) = 0$ or $(\beta, \alpha_k) = 0$ and $(\beta', \alpha_k) = \nu_\beta$. Thus $(\beta, \vartheta - \alpha_k) \neq 0$ is equivalent to $(\beta, \alpha_k) = 0$. This also gives that the number of such β in $\lambda(s_\vartheta)$ is $(l(s_\vartheta) - 1)/2$ because exactly one root from each pair $\{\beta, \beta'\}$ is orthogonal to α_k .

1.4. Main definition. Let m denote the least natural number such that $(P, P) = (1/m)\mathbb{Z}$. Thus $m = 2$ for D_{2k} , $m = 1$ for B_{2k} and C_k , and $m = |\Pi|$ otherwise.

The double affine Hecke algebra depends on the parameters q, t_ν ($\nu \in \nu_R$). It will be defined over the ring $\mathbb{Q}_{q,t} \stackrel{\text{def}}{=} \mathbb{Q}[q^{\pm 1/(2m)}, t_\nu^{\pm 1/2}]$. Later we will need the field of fractions $\mathbb{Q}'_{q,t} \stackrel{\text{def}}{=} \mathbb{Q}(q^{\pm 1/(2m)}, t_\nu^{1/2})$ and its subrings

$$(1.34) \quad \ddot{\mathbb{Q}}'_{q,t} \stackrel{\text{def}}{=} \{c \in \mathbb{Q}'_{q,t}; c \text{ is well defined when all } t_\nu^{1/2} = 0\},$$

$$(1.35) \quad \ddot{\mathbb{Q}}_{q,t}^\dagger \stackrel{\text{def}}{=} \{c \in \mathbb{Q}'_{q,t}; c \text{ is well defined when all } t_\nu^{-1/2} = 0\}.$$

We set

$$(1.36) \quad \begin{aligned} t_{\tilde{\alpha}} &= t_\alpha = t_{\nu_\alpha}, \quad t_i = t_{\alpha_i}, \quad q_{\tilde{\alpha}} = q^{\nu_\alpha}, \quad q_i = q^{\nu_{\alpha_i}}, \\ \text{where } \tilde{\alpha} &= [\alpha, \nu_{\alpha j}] \in \tilde{R}, \quad 0 \leq i \leq n. \end{aligned}$$

It will be convenient to use parameters $\{k_\nu\}$ together with $\{t_\nu\}$, setting here and further:

$$t_\alpha = t_\nu = q_\alpha^{k_\nu} \quad \text{for } \nu = \nu_\alpha, \quad \text{and } \rho_k = (1/2) \sum_{\alpha > 0} k_\alpha \alpha.$$

Note that $(\rho_k, \alpha_i^\vee) = k_i = k_{\alpha_i} = (\rho_k^\vee, \alpha_i)$ for $i > 0$, where

$$\rho_k^\vee \stackrel{\text{def}}{=} \sum k_\nu \rho_\nu^\vee \quad \text{for } \rho_\nu^\vee \stackrel{\text{def}}{=} \rho_\nu / \nu.$$

Using that $w_0(\rho_k) = -\rho_k$, we obtain that $(\rho_k, -w_0(b)) = (\rho_k, b)$. For instance, $(\rho_k, b_+) = -(\rho_k, b_-)$, where $b_+ \stackrel{\text{def}}{=} w_0(b_-)$ (see above).

By $q^{(\rho_k, \alpha)}$, we mean $\prod_{\nu \in \nu_R} t_\nu^{(\rho_\nu^\vee, \alpha)}$; here $\alpha \in R$, $(\rho_\nu^\vee, \alpha) \in \mathbb{Z}$ and this product contains only *integral* powers of t_{sht} and t_{lng} (non-negative if $\alpha > 0$).

For pairwise commutative X_1, \dots, X_n , let

$$(1.37) \quad X_{\tilde{b}} = \prod_{i=1}^n X_i^{l_i} q^j \quad \text{if } \tilde{b} = [b, j], \quad \widehat{w}(X_{\tilde{b}}) = X_{\widehat{w}(\tilde{b})}.$$

$$\text{where } b = \sum_{i=1}^n l_i \omega_i \in P, \quad j \in \frac{1}{m} \mathbb{Z}, \quad \widehat{w} \in \widehat{W}.$$

For instance, $X_0 \stackrel{\text{def}}{=} X_{\alpha_0} = qX_{\vartheta}^{-1}$.

We set $(\tilde{b}, \tilde{c}) = (b, c)$, ignoring the affine extensions in this pairing.

Recall that m_{ij} denotes the order of $s_i s_j$ in \widetilde{W} ($0 \leq i, j \leq n$) and that $r, r^* \in O'$ are related by $-w_0(\omega_r) = \omega_{r^*}$.

Definition 1.5. *The double affine Hecke algebra \mathcal{H} is generated over $\mathbb{Q}_{q,t}$ by the elements $\{T_i, 0 \leq i \leq n\}$, pairwise commutative $\{X_b, b \in P\}$ satisfying (1.37), and the group Π , where the following relations are imposed:*

- (o) $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0, \quad 0 \leq i \leq n;$
- (i) $T_i T_j T_i \cdots = T_j T_i T_j \cdots, m_{ij}$ factors on each side, $0 \leq i \neq j \leq n;$
- (ii) $\pi_r T_i \pi_r^{-1} = T_j$ if $\pi_r(\alpha_i) = \alpha_j;$
- (iii) $T_i X_b = X_b X_{\alpha_i}^{-1} T_i^{-1}$ if $(b, \alpha_i^\vee) = 1, \quad 0 \leq i \leq n;$
- (iv) $T_i X_b = X_b T_i$ if $(b, \alpha_i^\vee) = 0, \quad 0 \leq i \leq n;$
- (v) $\pi_r X_b \pi_r^{-1} = X_{\pi_r(b)} = X_{u_r^{-1}(b)} q^{(\omega_{r^*}, b)}, \quad r \in O'.$

One can rewrite (iii, iv) as in [L]:

$$(1.38) \quad T_i X_b - X_{s_i(b)} T_i = (t_i^{1/2} - t_i^{-1/2}) \frac{X_{s_i(b)} - X_b}{X_{\alpha_i} - 1}, \quad 0 \leq i \leq n.$$

Given $\tilde{w} \in \widetilde{W}, r \in O$, the product

$$(1.39) \quad T_{\pi_r \tilde{w}} \stackrel{\text{def}}{=} \pi_r \prod_{k=1}^l T_{i_k}, \quad \text{where } \tilde{w} = \prod_{k=1}^l s_{i_k}, l = l(\tilde{w}),$$

does not depend on the choice of the reduced decomposition (because T_i satisfy the same “braid” relations as s_i do). Moreover,

$$(1.40) \quad T_{\widehat{v}} T_{\widehat{w}} = T_{\widehat{v}\widehat{w}} \quad \text{whenever } l(\widehat{v}\widehat{w}) = l(\widehat{v}) + l(\widehat{w}) \quad \text{for } \widehat{v}, \widehat{w} \in \widehat{W}.$$

In particular, we arrive at the pairwise commutative elements:

$$(1.41) \quad Y_b = \prod_{i=1}^n Y_i^{l_i} \quad \text{if } b = \sum_{i=1}^n l_i \omega_i \in P, \quad Y_i \stackrel{\text{def}}{=} T_{\omega_i}, \quad b \in P.$$

For any $b \in P$, the element Y_b can be presented as the product $\pi_r T_{j_l}^{\pm 1} \cdots T_{j_1}^{\pm 1}$ for any reduced decomposition $b = \pi_r s_{j_l} \cdots s_{j_1}$ and a proper choice of signs \pm (see (1.42) below). Note that $l = l(b) = 2(\rho^\vee, b_+)$ depends only on b_+ . The total number of factors $T_j^{\pm 1}$ in this product with $\nu_j = \nu$ equals $2(\rho_\nu^\vee, b_+)$.

The signs \pm can be described as follows (see [M3, (3.2.10)]). Given the reduced decomposition above, form $\lambda(b)$ using (1.15) and write $\tilde{\alpha}^p = [\alpha^p, \nu_{\alpha^p} j]$. Then one has $Y_b = \pi_r T_{j_l}^{\epsilon_l} \cdots T_{j_1}^{\epsilon_1}$, where

$$(1.42) \quad \epsilon_p = \begin{cases} +1 & \text{if } \alpha^p > 0, \\ -1 & \text{if } \alpha^p < 0. \end{cases}$$

Duality anti-involution. There exists a unique anti-involution φ of \mathcal{H} satisfying (see [C2]):

$$(1.43) \quad \varphi : X_b \leftrightarrow Y_{-b}, \quad T_i \mapsto T_i \quad (1 \leq i \leq n), \quad q^{\frac{1}{2m}} \mapsto q^{\frac{1}{2m}}, \quad t_\nu^{1/2} \mapsto t_\nu^{1/2}.$$

Using $Y_\vartheta = T_0 T_{s_\vartheta}$ and $Y_{\omega_r} = \pi_r T_{u_r}$, one finds that

$$(1.44) \quad \varphi(T_0) = T_{s_\vartheta}^{-1} X_\vartheta^{-1}, \quad \varphi(\pi_r) = T_{u_r^{-1}}^{-1} X_{\omega_r}^{-1} = X_{\omega_r^*} T_{u_r} = \varphi(\pi_{r^*}^{-1}).$$

Applying φ to (iii, iv) in the definition of \mathcal{H} , we obtain the dual relations (for $i > 0$ only):

$$(1.45) \quad \begin{aligned} T_i^{-1} Y_b &= Y_{s_i(b)} T_i \quad \text{if } (b, \alpha_i^\vee) = 1, \\ T_i Y_b &= Y_b T_i \quad \text{if } (b, \alpha_i^\vee) = 0, \quad 1 \leq i \leq n. \end{aligned}$$

The counterpart of (1.38) reads as follows:

$$(1.46) \quad T_i Y_b - Y_{s_i(b)} T_i = (t_i^{1/2} - t_i^{-1/2}) \frac{Y_b - Y_{s_i(b)}}{1 - Y_{-\alpha_i}}, \quad 1 \leq i \leq n.$$

Automorphisms. We will need the following automorphisms of \mathcal{H} . We refer to [C8] and the references therein for proofs and for a discussion of how these automorphisms can be described in terms of an action of $PSL(2, \mathbb{Z})$.

We say that an automorphism (or anti-automorphism) of \mathcal{H} *preserves* q, t_ν if it fixes all fractional powers of these parameters (i.e., it is $\mathbb{Q}_{q,t}$ -linear). We say that an automorphism *conjugates* q, t_ν to mean that it sends all fractional powers of these parameters to their inverses (so such a map is only \mathbb{Q} -linear).

The following map can be uniquely extended to an automorphism of \mathcal{H} fixing T_i ($i \geq 1$) and preserving q, t_ν :

$$(1.47) \quad \begin{aligned} \tau_+ : X_b &\mapsto X_b, \quad \pi_r \mapsto q^{-\frac{(\omega_r, \omega_r)}{2}} X_{\omega_r} \pi_r, \quad Y_{\omega_r} \mapsto X_{\omega_r} Y_{\omega_r} q^{-\frac{(\omega_r, \omega_r)}{2}}, \\ T_0 &\mapsto X_{\alpha_0}^{-1} T_0^{-1}, \quad Y_\vartheta \mapsto X_{\alpha_0}^{-1} T_0^{-1} T_{s_\vartheta}. \end{aligned}$$

Define the automorphism $\tau_- \stackrel{\text{def}}{=} \varphi \tau_+ \varphi$. Explicitly, τ_- fixes T_i ($i \geq 1$), as well as τ_+ , preserves q, t_ν , and satisfies

$$(1.48) \quad \tau_- : X_{\omega_r} \mapsto q^{(\omega_r, \omega_r)/2} Y_{\omega_r} X_{\omega_r}, \quad \pi_r \mapsto \pi_r, \quad T_0 \mapsto T_0, \quad Y_b \mapsto Y_b.$$

We also need the following automorphism of \mathcal{H} :

$$(1.49) \quad \sigma \stackrel{\text{def}}{=} \tau_+ \tau_-^{-1} \tau_+ = \tau_-^{-1} \tau_+ \tau_-^{-1};$$

it preserves q, t_ν and satisfies

$$(1.50) \quad \begin{aligned} \sigma : T_i &\mapsto T_i \quad (i > 0), \quad X_b \mapsto Y_b^{-1}, \quad Y_{\omega_r} \mapsto q^{-(\omega_r, \omega_r)} Y_{\omega_r}^{-1} X_{\omega_r} Y_{\omega_r}, \\ \pi_r &\mapsto X_{\omega_r} T_{u_r^{-1}} = T_{u_r}^{-1} X_{\omega_r}^{-1}, \quad T_0 \mapsto T_{s_\vartheta}^{-1} X_\vartheta^{-1}. \end{aligned}$$

The equality of the two expressions for $\sigma(\pi_r)$ follows from (1.44).

The following map can be uniquely extended to an involution of \mathcal{H} conjugating q, t_ν :

$$(1.51) \quad \varepsilon : T_i \mapsto T_i^{-1} \quad (i > 0), \quad X_b \leftrightarrow Y_b.$$

Using $Y_\vartheta = T_0 T_{s_\vartheta}$ and $Y_{\omega_r} = \pi_r T_{u_r}$, one finds that

$$(1.52) \quad \varepsilon(T_0) = X_\vartheta T_{s_\vartheta}, \quad \varepsilon(\pi_r) = X_{\omega_r} T_{u_r}^{-1}.$$

We will also need the involution $\eta \stackrel{\text{def}}{=} \varepsilon \sigma = \sigma^{-1} \varepsilon$. Explicitly, η conjugates q, t_ν and satisfies

$$(1.53) \quad \eta : T_i \mapsto T_i^{-1} \quad (i \geq 0), \quad X_b \mapsto X_b^{-1}, \quad \pi_r \mapsto \pi_r.$$

1.5. Double-dot normalization. The case when all $t_\nu = 0$ will play an important role in this paper. Definition 1.5, which matches that from [C8], [C4], and other first author's papers, is not suited to this specialization. We introduce the following normalization to handle the specialization $t_\nu = 0$.

We set

$$(1.54) \quad \ddot{T}_i \stackrel{\text{def}}{=} t_i^{1/2} T_i, \quad \ddot{T}_i' \stackrel{\text{def}}{=} t_i^{1/2} T_i^{-1} = \ddot{T}_i - (t_i - 1).$$

Note that the same normalization is used for both T_i and T_i^{-1} , so that $\ddot{T}_i \ddot{T}_i' = t_i$. Thus $(\ddot{T}_i)^{-1}$ and \ddot{T}_i^{-1} do not coincide.

We observe that $\{\ddot{T}_i\}$ satisfy the same braid relations as $\{T_i\}$. The quadratic relations read: $(\ddot{T}_i - t_i)(\ddot{T}_i + 1) = 0$.

The dot-normalization $\{\ddot{\cdot}\}$ will be applied term-wise to the products of $\pi_r \in \Pi, T_i, T_i^{-1}$ provided that the corresponding word in \widehat{W} is reduced. We set $\ddot{\pi}_r = \pi_r$.

Thus, using the description of Y_b from the previous section, one has $\ddot{Y}_b = q^{(b_+, \rho_k)} Y_b$. Equivalently, one can set $\ddot{Y}_i \stackrel{\text{def}}{=} \ddot{T}_{\omega_i}$ and define

$$(1.55) \quad \ddot{Y}_b = q^{(b_+ - b, \rho_k)} \prod_{i=1}^n \ddot{Y}_i^{l_i} \quad \text{if } b = \sum_{i=1}^n l_i \omega_i.$$

Note that $\ddot{Y}_b \ddot{Y}_{-b} = q^{2(b_+, \rho_k)}$.

The first line in (1.45) can be rewritten as

$$(1.56) \quad \ddot{T}'_i \ddot{Y}_b = \ddot{Y}_{s_i(b)} \ddot{T}_i \quad \text{if } (b, \alpha_i^\vee) = 1,$$

and (1.46) becomes

$$(1.57) \quad \ddot{T}_i \ddot{Y}_b - \ddot{Y}_{s_i(b)} \ddot{T}_i = (t_i - 1) \frac{\ddot{Y}_b - \ddot{Y}_{s_i(b)}}{1 - q^{-(\theta_i, \rho_k)} \ddot{Y}_{-\alpha_i}}, \quad 1 \leq i \leq n,$$

where $\theta_i = \theta, \vartheta$ for long, short α_i respectively; to see this, use that θ_i is the only root in the intersection $W(\alpha_i) \cap P_+$.

We come to the following definition. Let us extend the scalars of \mathcal{H} to $\mathbb{Q}'_{q,t}$ and define $\ddot{\mathcal{H}} \subset \mathcal{H}$ to be the subalgebra generated over $\mathbb{Q}'_{q,t}$ by the elements

$$X_a \ (a \in P), \quad \ddot{T}_{\widehat{w}} \ (\widehat{w} \in \widehat{W}), \quad \ddot{Y}_b \ (b \in P).$$

It suffices to take here only

$$X_a \ (a \in P), \quad \ddot{T}_i \ (i \geq 0), \quad \Pi.$$

Definition 1.6. *The defining relations of $\ddot{\mathcal{H}}$ as an abstract algebra are as follows:*

- (o) $(\ddot{T}_i - t_i)(\ddot{T}_i + 1) = 0, \ 0 \leq i \leq n;$
- (i) $\ddot{T}_i \ddot{T}_j \ddot{T}_i \cdots = \ddot{T}_j \ddot{T}_i \ddot{T}_j \cdots, m_{ij} \text{ factors on each side}, \ 0 \leq i \neq j \leq n;$
- (ii) $\pi_r \ddot{T}_i \pi_r^{-1} = \ddot{T}_j \quad \text{if } \pi_r(\alpha_i) = \alpha_j, \ \pi_r \in \Pi;$
- (iii) $\ddot{T}_i X_b = X_b X_{\alpha_i}^{-1} \ddot{T}'_i \quad \text{if } (b, \alpha_i^\vee) = 1, \ 0 \leq i \leq n;$
- (iv) $\ddot{T}_i X_b = X_b \ddot{T}_i \quad \text{if } (b, \alpha_i^\vee) = 0 \text{ for } 0 \leq i \leq n;$
- (v) $\pi_r X_b \pi_r^{-1} = X_{\pi_r(b)} = X_{u_r^{-1}(b)} q^{(\omega_{r^*}, b)}.$

□

The limit (reduction) of $\dot{\mathcal{H}}$ when $t_\nu = 0$ for all or some ν will be denoted by $\overline{\mathcal{H}}$ later and will be called the *nil-DAHA*.

2. POLYNOMIAL REPRESENTATION

From now on, we will switch from \mathcal{H} to its *intermediate subalgebra* $\mathcal{H}^b \subset \mathcal{H}$ generated by T_i ($i \geq 0$), X_a ($a \in B$), and Y_a ($a \in B$), where B is any lattice between Q and P (see [C7]). Accordingly, we replace Π by the preimage Π^b of B/Q in Π . Generally, there can be two different lattices B_X and B_Y for X and Y . We consider only $B_X = B = B_Y$ in the paper. It is straightforward to check that all (anti-)automorphisms introduced in the previous section preserve \mathcal{H}^b . The $\ddot{\mathbb{Q}}'_{q,t}$ -subalgebra $\dot{\mathcal{H}}^b \subset \mathcal{H}^b$ is defined accordingly.

We also set $\widehat{W}^b = B \cdot W \subset \widehat{W}$ and replace m by the least $\tilde{m} \in \mathbb{N}$ such that $\tilde{m}(B, B) \subset \mathbb{Z}$ in the definition of $\mathbb{Q}_{q,t}$, $\mathbb{Q}'_{q,t}$, $\ddot{\mathbb{Q}}'_{q,t}$, and $\ddot{\mathbb{Q}}^\dagger_{q,t}$.

We point out that \mathcal{H}^b and the polynomial representation can be defined over the ring $\mathbb{Z}[q^{\pm 1/(2\tilde{m})}, t_\nu^{\pm 1/2}]$. However, the ring $\mathbb{Q}_{q,t}$ and its localizations above will be sufficient in this paper.

The *Demazure-Lusztig operators* are

$$(2.1) \quad T_i = t_i^{1/2} s_i + (t_i^{1/2} - t_i^{-1/2})(X_{\alpha_i} - 1)^{-1}(s_i - 1), \quad 0 \leq i \leq n;$$

they obviously preserve $\mathbb{Q}_{q,t}[X] \stackrel{\text{def}}{=} \mathbb{Q}_{q,t}[X_b, b \in B]$. We note that only the formula for T_0 involves q :

$$(2.2) \quad \begin{aligned} T_0 &= t_0^{1/2} s_0 + (t_0^{1/2} - t_0^{-1/2})(X_0 - 1)^{-1}(s_0 - 1), \text{ where} \\ X_0 &= qX_\vartheta^{-1}, \quad s_0(X_b) = X_b X_\vartheta^{-(b, \vartheta)} q^{(b, \vartheta)}, \quad \alpha_0 = [-\vartheta, 1]. \end{aligned}$$

The map sending T_i ($i \geq 0$) to the corresponding operator from (2.1), X_b to X_b (see (1.37)) and $\pi_r \mapsto \pi_r$ induces a $\mathbb{Q}_{q,t}$ -linear homomorphism from \mathcal{H}^b to the algebra of linear endomorphisms of $\mathbb{Q}_{q,t}[X]$. We will extend the ring of constants here to the field of fractions $\mathbb{Q}'_{q,t}$ or its subring $\ddot{\mathbb{Q}}'_{q,t}$ of the rationals well defined when $t_\nu = 0$ for all ν .

This \mathcal{H}^b -module is faithful and remains faithful when q, t take any complex values assuming that $q \neq 0$ is not a root of unity. It will be called the *polynomial representation*; the notation is $\mathcal{V} \stackrel{\text{def}}{=} \mathbb{Q}'_{q,t}[X]$. We also set $\ddot{\mathcal{V}} \stackrel{\text{def}}{=} \ddot{\mathbb{Q}}'_{q,t}[X]$ and $\ddot{\mathcal{V}}^\dagger \stackrel{\text{def}}{=} \ddot{\mathbb{Q}}^\dagger_{q,t}[X]$.

Given any $H \in \mathcal{H}^b$, we continue to denote by H the corresponding operator in \mathcal{V} .

The polynomial representation can be described as the \mathcal{H}^b -module induced from the one-dimensional representation $T_i \mapsto t_i^{1/2}$ ($i \geq 0$), $Y_b \mapsto q^{(\rho_k, b)}$ of the affine Hecke subalgebra $\mathcal{H}_Y^b = \langle T_i, Y_b \rangle$.

Elements of \mathcal{H}^b act in \mathcal{V} by *difference-reflection operators*, which are operators of the form

$$(2.3) \quad \sum_{w \in W, b \in B} g_{b,w} \Gamma_b w, \quad g_{b,w} \in \mathbb{Q}'_{q,t}(X),$$

where $\mathbb{Q}'_{q,t}(X)$ is the field of rational functions in the X_b ($b \in B$). We denote the algebra of all such operators by \mathcal{A} ; its defining relations are as follows:

$$(2.4) \quad q^{(a,b)} X_a \Gamma_b = \Gamma_b X_a, \quad w X_a = X_{w(a)} w, \quad w \Gamma_b = \Gamma_{w(b)} w.$$

The algebra of *difference operators* is the subalgebra of \mathcal{A} generated by $\mathbb{Q}'_{q,t}(X)$ and Γ_b ($b \in B$). There is a natural linear map

$$(2.5) \quad \text{Red} : \sum_{w \in W, b \in B} g_{b,w}(X) \Gamma_b w \mapsto \sum_{w \in W, b \in B} g_{b,w} \Gamma_b,$$

sending difference-reflection operators to difference operators. Clearly, Red is *not* a homomorphism of algebras.

The images of the Y_b in the polynomial representation are called the *difference Dunkl operators*. Later we will make use of the following explicit description of these operators. Let $b = \pi_r s_{j_l} \cdots s_{j_1}$ be a reduced decomposition of any $b \in B$, and recall the definition of ϵ_p from (1.42). Then

$$(2.6) \quad Y_b = \pi_r T_{j_l}^{\epsilon_l} \cdots T_{j_1}^{\epsilon_1} = b G_{\tilde{\alpha}^l}^{\text{sgn}(\epsilon_l)} \cdots G_{\tilde{\alpha}^1}^{\text{sgn}(\epsilon_1)},$$

where $\text{sgn}(\pm 1) = \pm$ and

$$(2.7) \quad G_{\tilde{\alpha}}^+ \stackrel{\text{def}}{=} t_{\alpha}^{1/2} + \frac{t_{\alpha}^{1/2} - t_{\alpha}^{-1/2}}{X_{\tilde{\alpha}}^{-1} - 1} (1 - s_{\tilde{\alpha}}) = t_{\alpha}^{-1/2} (f_{\tilde{\alpha}} + g_{\tilde{\alpha}} s_{\tilde{\alpha}}),$$

$$f_{\tilde{\alpha}} = \frac{t_{\alpha} X_{\tilde{\alpha}}^{-1} - 1}{X_{\tilde{\alpha}}^{-1} - 1}, \quad g_{\tilde{\alpha}} = \frac{t_{\alpha} - 1}{1 - X_{\tilde{\alpha}}^{-1}};$$

$$(2.8) \quad G_{\tilde{\alpha}}^- \stackrel{\text{def}}{=} t_{\alpha}^{-1/2} + \frac{t_{\alpha}^{1/2} - t_{\alpha}^{-1/2}}{1 - X_{\tilde{\alpha}}} (1 - s_{\tilde{\alpha}}) = t_{\alpha}^{-1/2} (f_{\tilde{\alpha}} - s_{\tilde{\alpha}} g_{\tilde{\alpha}}).$$

Note that

$$G_{\alpha_i}^+ = s_i T_i, \quad G_{-\alpha_i}^+ = T_i s_i, \quad G_{\alpha_i}^- = s_i T_i^{-1}, \quad \text{and} \quad G_{-\alpha_i}^- = T_i^{-1} s_i.$$

Let $\ddot{G}_{\tilde{\alpha}}^{\pm} \stackrel{\text{def}}{=} t_{\tilde{\alpha}}^{1/2} G_{\tilde{\alpha}}^{\pm}$, so that $\ddot{Y}_b = b \ddot{G}_{\tilde{\alpha}^l}^{\text{sgn}(\epsilon_l)} \cdots \ddot{G}_{\tilde{\alpha}^1}^{\text{sgn}(\epsilon_1)}$.

2.1. Macdonald polynomials. There are two equivalent definitions of the *nonsymmetric Macdonald polynomials*, denoted $E_b(X) = E_b^{(k)}$ for $b \in B$.

The first definition is based on the *truncated theta function*:

$$(2.9) \quad \mu(X; t) = \mu^{(k)}(X) = \prod_{\alpha \in R_+} \prod_{j=0}^{\infty} \frac{(1 - X_{\alpha} q_{\alpha}^j)(1 - X_{\alpha}^{-1} q_{\alpha}^{j+1})}{(1 - X_{\alpha} t_{\alpha} q_{\alpha}^j)(1 - X_{\alpha}^{-1} t_{\alpha} q_{\alpha}^{j+1})}.$$

We will mainly consider μ as a Laurent series with coefficients in the ring $\mathbb{Q}[t_{\nu}][[q_{\nu}]]$ for $\nu \in \nu_R = \{\nu_{\text{sht}}, \nu_{\text{lng}}\}$. The constant term of a Laurent series $f(X)$ will be denoted by $\langle f \rangle$. One has

$$(2.10) \quad \langle \mu \rangle = \prod_{\alpha \in R_+} \prod_{j=1}^{\infty} \frac{(1 - q^{(\rho_k, \alpha) + j \nu_{\alpha}})^2}{(1 - t_{\alpha} q^{(\rho_k, \alpha) + j \nu_{\alpha}})(1 - t_{\alpha}^{-1} q^{(\rho_k, \alpha) + j \nu_{\alpha}})}.$$

This equality is equivalent to the Macdonald constant term conjecture proved in complete generality in [C3].

Let $\mu_{\circ} \stackrel{\text{def}}{=} \mu / \langle \mu \rangle$. The coefficients of the Laurent series μ_{\circ} belong to $\mathbb{Q}(q, t_{\nu})$ for $\nu \in \nu_R$ and are well defined at $t_{\nu} = 0$. Define an inner product on \mathcal{V} by

$$(2.11) \quad \langle f, g \rangle_{\circ} \stackrel{\text{def}}{=} \langle f g^{\star} \mu_{\circ} \rangle,$$

where \star is the \mathbb{Q} -linear involution on \mathcal{V} defined by

$$(2.12) \quad X_b^{\star} = X_b^{-1}, \quad (q^{1/(2m)})^{\star} = q^{-1/(2m)}, \quad (t_{\nu}^{1/2})^{\star} = t_{\nu}^{-1/2}.$$

One has $\mu_{\circ}^{\star} = \mu_{\circ}$ and consequently $\langle g, f \rangle = \langle f, g \rangle^{\star}$.

The polynomials E_b are uniquely determined from the relations

$$(2.13) \quad E_b - X_b \in \oplus_{c \succ b} \mathbb{Q}'_{q,t} X_c, \quad \langle E_b, X_c \rangle = 0 \quad \text{for} \quad B \ni c \succ b$$

and for generic q, t ; they form a basis of \mathcal{V} . Their coefficients actually belong to $\mathbb{Q}(q, t_{\nu})$ and are well defined at $t_{\nu} = 0$; for the latter, see (2.30) below.

This definition is due to Macdonald (for $k_{\text{sht}} = k_{\text{lng}} \in \mathbb{Z}_+$), who extended the construction from [Op]. The general (reduced) case was considered in [C4].

We note that the E_b satisfy the stronger condition (see [M3, (2.7.5)]):

$$(2.14) \quad E_b - X_b \in \oplus_{c \succ b} \mathbb{Q}'_{q,t} X_c,$$

in terms of the partial ordering \succ defined in (1.21).

The second definition of the E -polynomials is based on the Dunkl operators:

Proposition 2.1. *The polynomials $\{E_b, b \in B\}$ are the unique (up to proportionality) eigenfunctions of the operators Y_a ($a \in B$) acting in \mathcal{V} :*

$$(2.15) \quad Y_a(E_b) = q^{-(a, b_\sharp)} E_b \text{ for } b_\sharp \stackrel{\text{def}}{=} b - u_b^{-1}(\rho_k),$$

$$\text{equivalently, } \ddot{Y}_a(E_b) = q^{(a_+, \rho_k) - (a, b_\sharp)} E_b,$$

where $u_b = \pi_b^{-1}b$ is from Proposition 1.2, $b_\sharp = \pi_b((-\rho_k))$. \square

The second definition readily leads to the orthogonality of the E -polynomials. This is due to the fact that

$$(2.16) \quad \langle H(f), g \rangle = \langle f, H^*(g) \rangle, \text{ for } f, g \in \mathcal{V}, H \in \mathcal{H}^b,$$

where \star is the anti-involution of \mathcal{H}^b extending (2.12) and defined by

$$(2.17) \quad \star : T_i \mapsto T_i^{-1} \ (i > 0), \ X_b \mapsto X_b^{-1}, \ Y_b \mapsto Y_b^{-1}, \ \pi_r \mapsto \pi_r^{-1}.$$

The norms of the E -polynomials are given explicitly by

$$(2.18) \quad \langle E_b, E_c \rangle_\circ = \delta_{bc} \prod_{[\alpha, j] \in \lambda'(\pi_b)} \frac{(1 - q_\alpha^j t_\alpha^{-1} X_\alpha(q^{\rho_k}))(1 - q_\alpha^j t_\alpha X_\alpha(q^{\rho_k}))}{(1 - q_\alpha^j X_\alpha(q^{\rho_k}))(1 - q_\alpha^j X_\alpha(q^{\rho_k}))},$$

where we set $X_a(q^z) = q^{(a, z)}$ for $a \in B$; see [C4].

We note that

$$(2.19) \quad (H(f))^* = \eta(H)(f^*), \text{ for } f \in \mathcal{V}, H \in \mathcal{H}^b,$$

where η is the involution defined in (1.53).

2.2. Symmetric polynomials. For $f \in \mathbb{Q}'_{q,t}[X]^W$, let

$$(2.20) \quad \mathcal{L}_f \stackrel{\text{def}}{=} f(Y_{\omega_1}, \dots, Y_{\omega_n}) = \sum_{w \in W, b \in P} g_{b,w} \Gamma_b w, \quad g_{b,w} \in \mathbb{Q}'_{q,t}(X),$$

$$L_f \stackrel{\text{def}}{=} \text{Red}(\mathcal{L}_f) = \sum_{w \in W, b \in P} g_{b,w} \Gamma_b.$$

The operators \mathcal{L}_f and L_f preserve \mathcal{V}^W and they coincide upon the restriction to this space. Moreover, the L_f are W -invariant difference operators, i.e., $wL_fw^{-1} = L_f$ for any $w \in W$. If f has coefficients in $\mathbb{Q}'_{q,t}$, we set $\check{\mathcal{L}}_f \stackrel{\text{def}}{=} f(\check{Y}_{\omega_1}, \dots, \check{Y}_{\omega_n})$ and $\check{L}_f = \text{Red}(\check{\mathcal{L}}_f)$.

Following Proposition 2.1, the *symmetric Macdonald polynomials* $P_b = P_b^{(k)}$ can be introduced as eigenfunctions of these operators. Explicitly,

$$(2.21) \quad \begin{aligned} L_f(P_{b_-}) &= f(q^{-b_- + \rho_k}) P_{b_-}, \quad b_- \in B_-, \\ P_{b_-} &= \sum_{b \in W(b_-)} X_b \mod \oplus_{c \succ b_-} \mathbb{Q}'_{q,t} X_c. \end{aligned}$$

These polynomials were introduced in [M1]; For classical root systems, they were first used in an unpublished work of Kadell. In the case of A_1 , they are due to Rogers.

For $a \in B$, let $\mathcal{L}_a \stackrel{\text{def}}{=} \mathcal{L}_f$ and $L_a \stackrel{\text{def}}{=} L_f$ where $f = \sum_{w \in W/W^a} Y_{w(a)}$; recall that W^a is the stabilizer of a in W . For these operators, (2.21) reads

$$(2.22) \quad \begin{aligned} L_a(P_{b_-}) &= \left(\sum_{a' \in W(a)} q^{-(a', b_- - \rho_k)} \right) P_{b_-}, \\ \check{L}_a(P_{b_-}) &= q^{(a_+, \rho_k)} \left(\sum_{a' \in W(a)} q^{-(a', b_- - \rho_k)} \right) P_{b_-}. \end{aligned}$$

The connection between E and P is as follows:

$$(2.23) \quad \begin{aligned} P_{b_-} &= \mathbf{P}_{b_+} E_{b_+}, \quad b_- \in B_-, \quad b_+ = w_0(b_-), \\ \mathbf{P}_{b_+} &\stackrel{\text{def}}{=} \sum_{c \in W(b_+)} \check{T}_{w_c} = \sum_{c \in W(b_+)} t^{l(w_c)/2} T_{w_c}, \\ t^{l(\hat{w})} &\stackrel{\text{def}}{=} \prod_{\nu} t_{\nu}^{l_{\nu}(\hat{w})} \quad \text{for } l_{\nu}(\hat{w}) = |\{\tilde{\alpha} \in \lambda(\hat{w}), \nu_{\alpha} = \nu\}|, \end{aligned}$$

where $w_c \in W$ is the element of least length such that $c = w_c(b_+)$. Hence P_{b_-} belongs to $\mathbb{Q}'_{q,t}[X]$. Taking the complete t -symmetrization \mathbf{P} here (with the summation over all w), one obtains P_{b_-} up to proportionality. See [Op, M2, C4].

There are two different kinds of inner products in \mathcal{V} from [C8] and other works, with and without using \star ; we will mainly need the former in this work, which is \langle, \rangle_{\circ} from (2.11). In the symmetric setting, they

essentially coincide. The inner products of the symmetric polynomials P_b for $b = b_-$ read as follows (see [C8]):

$$(2.24) \quad \langle P_b, P_c \rangle_\circ = \delta_{bc} \prod_{\alpha > 0} \prod_{j=0}^{-(\alpha^\vee, b)-1} \frac{(1 - q_\alpha^{j+1} t_\alpha^{-1} X_\alpha(q^{\rho_k}))(1 - q_\alpha^j t_\alpha X_\alpha(q^{\rho_k}))}{(1 - q_\alpha^j X_\alpha(q^{\rho_k}))(1 - q_\alpha^{j+1} X_\alpha(q^{\rho_k}))}.$$

2.3. Using intertwiners. The Y -intertwiners serve as creation operators in the theory of nonsymmetric Macdonald polynomials. Let $0 \leq i \leq n$, $Y_0 = Y_{\alpha_0} \stackrel{\text{def}}{=} q^{-1} Y_{\vartheta}^{-1}$. Following [C8] here and below, we set

$$(2.25) \quad \Psi_i = \tau_+(T_i) + \frac{t_i^{1/2} - t_i^{-1/2}}{Y_{\alpha_i}^{-1} - 1}, \quad \Psi_i^b = \tau_+(T_i) + \frac{t_i^{1/2} - t_i^{-1/2}}{X_{\alpha_i}(q^{b_\sharp}) - 1}.$$

Recall the definition of the automorphism τ_+ of \mathcal{H}^b from (1.47). In the following theorem, which explicitly describes the action of the intertwiners on the E -polynomials, we also need the pairing from (1.8) and the affine action $\hat{w}((b))$ from (1.7).

Theorem 2.2. *Given $b \in B$, $0 \leq i \leq n$ such that $(\alpha_i, b + d) > 0$,*

$$(2.26) \quad q^{-\frac{(c,c)}{2}} E_c = t_i^{\frac{1}{2}} \Psi_i^b(q^{-\frac{(b,b)}{2}} E_b), \quad \frac{q^{-\frac{(b,b)}{2}} E_b}{\langle E_b, E_b \rangle_\circ} = t_i^{-\frac{1}{2}} \Psi_i^c\left(\frac{q^{-\frac{(c,c)}{2}} E_c}{\langle E_c, E_c \rangle_\circ}\right),$$

where $c = s_i((b))$. If $(\alpha_i, b + d) = 0$, then

$$(2.27) \quad \tau_+(T_i)(E_b) = t_i^{1/2} E_b, \quad \tau_+(\ddot{T}_i)(E_b) = t_i E_b, \quad 0 \leq i \leq n,$$

which results in the relations $s_i(E_b) = E_b$ as $i > 0$. For $c = \pi_r((b))$, where the indices r are from O' ,

$$(2.28) \quad q^{(b,b)/2 - (c,c)/2} E_c = \tau_+(\pi_r)(E_b) = X_{\omega_r} q^{-(\omega_r, \omega_r)/2} \pi_r(E_b).$$

Also $\tau_+(\pi_r)(E_b) \neq E_b$ for $\pi_r \neq \text{id}$, since $\pi_r((b)) \neq b$ for any $b \in B$. \square

If $(\alpha_i, b) > 0$ and $i > 0$, then the set $\lambda(\pi_c)$ is obtained from $\lambda(\pi_b)$ by adding $[\alpha, (b_-, \alpha)]$ for $\alpha = u_b(\alpha_i) \in R_-$ and $(b_-, \alpha^\vee) = (b, \alpha_i^\vee) > 0$. When $i = 0$ and $(\alpha_0, b + d) = -(b, \vartheta) + 1 > 0$, then the root $[\alpha, (b_-, \alpha) + 1]$ is added to $\lambda(\pi_b)$ for $\alpha = u_b(-\vartheta) = \alpha^\vee \in R_-$ and $(b_-, \alpha) = -(b, \vartheta) \geq 0$.

In each of these two cases, $(\alpha_i, u_b^{-1}(\rho)) = (\alpha, \rho) < 0$ and the powers of t_ν in

$$(2.29) \quad X_{\alpha_i}(q^{b_i}) = q^{(\alpha_i, b - u_b^{-1}(\rho_k) + d)} = q^{(\alpha_i, b + d)} \prod_{\nu} t_{\nu}^{-(\alpha, \rho_{\nu}^{\vee})}$$

are from \mathbb{Z}_+ , with that of t_i strictly positive.

Due to Theorem 2.2 (see also [C6], Corollary 5.3), the coefficients of polynomial E_b belong to $\mathbb{Q}_{q,t}$ divided by

$$(2.30) \quad \prod_{[-\alpha, \nu_{\alpha} j] \in \lambda(\pi_b)} (1 - q_{\alpha}^j X_{\alpha}(q^{\rho_k})).$$

More exactly, the ring $\mathbb{Q}_{q,t}$ can be replaced here by $\mathbb{Q}[q, t_{\nu}]$, which readily results in the existence of the limits of the E -polynomials when $t_{\nu} = 0$ for all $\nu \in \nu_R$. Thus their coefficients become polynomials in terms of q in this limit.

Here the key is that powers of q appearing in (2.30) are always multiplied by nonzero powers of t_{ν} . The same argument and a relatively straightforward analysis of the leading t -powers of the coefficients of E_b can be used to see that the limits of E_b exist when $t_{\nu} \rightarrow \infty$. Moreover, their coefficients become polynomials in q^{-1} in this limit; see Corollary 2.6 below.

2.4. The limit $t \rightarrow 0$. The limit (reduction) of $\dot{\mathcal{H}}^b$ introduced in Definition 1.6 when $t_{\nu} = 0$ for all or some ν will be denoted by $\overline{\mathcal{H}}^b$ and called the *nil-DAHA*. For the sake of definiteness, *all $t_{\nu} = 0$ in this section*.

The polynomials \overline{E}_b , the images of E_b as $t_{\nu} = 0$ for $\nu \in \nu_R$ linearly generate the bar-polynomial representation:

$$\overline{\mathcal{V}} \stackrel{\text{def}}{=} \mathbb{Q}'_q[X_b, b \in B], \quad \text{where } \mathbb{Q}'_q \stackrel{\text{def}}{=} \mathbb{Q}(q^{1/(2m)}).$$

Thus, the $\overline{\mathcal{H}}^b$ action in $\overline{\mathcal{V}}$ is given by the operators

$$(2.31) \quad \overline{T}_i \stackrel{\text{def}}{=} \ddot{T}_i(t_i = 0) = (X_{\alpha_i} - 1)^{-1}(1 - s_i), \quad 0 \leq i \leq n,$$

$$(2.32) \quad \overline{Y}_a \stackrel{\text{def}}{=} \ddot{Y}_a(t_{\nu} = 0), \quad a \in B,$$

and the action of Π and multiplication by X_b ($b \in B$).

The \overline{E} -polynomials are eigenfunctions of the \overline{Y} -operators. Using (2.15), one has explicitly:

$$(2.33) \quad \overline{Y}_a(\overline{E}_b) = \begin{cases} q^{-(a,b)} \overline{E}_b, & \text{if } u_b(a) = a_-, \\ 0, & \text{otherwise.} \end{cases}$$

Note that using only $\overline{Y}_a(a \in B_+)$ is obviously not sufficient to split $\{\overline{E}_b\}$ for generic q ; all $a \in B$ must be involved.

Theorem 2.2 holds under this limit and gives quite a constructive approach to the \overline{E} -polynomials. The reductions $\overline{\Psi}_i^c$ of the intertwiners $\check{\Psi}_i^c \stackrel{\text{def}}{=} t_i^{1/2} \Psi_i^c$ from (2.25) can be used to generate \overline{E}_b ; these intertwiners become $\tau_+(\overline{T}_i) + 1$ in this limit.

This simplification is directly connected with the fact that

$$\overline{T}_i' \stackrel{\text{def}}{=} \check{T}_i'(t_i = 0) = \overline{T}_i + 1$$

satisfy the same homogeneous Coxeter relations as $\{T_i, 0 \leq i \leq n\}$ do, a special feature of the nil-DAHA. It readily results from the theory of intertwiners, and, of course, can be checked directly as well.

Let us provide some details of the construction of bar-polynomials. The action of π_r on $\{\overline{T}_i'\}$ by conjugation obviously remains unchanged. Thus relations (i,ii) from Definition 1.5 (and above) hold for $\{\overline{T}_i'\}$. Therefore, given $\hat{w} \in \widehat{W}$, the element $\overline{T}_{\hat{w}}' = \pi_r \overline{T}_{i_1}' \cdots \overline{T}_{i_l}'$ does not depend on the choice of the reduced decomposition $\hat{w} = \pi_r s_{i_l} \cdots s_{i_1}$.

For instance, the operators $\overline{\Pi}_i' \stackrel{\text{def}}{=} \tau_+(\overline{T}_{-\omega_i}')^n$ for $i = 1, \dots, n$ are pairwise commutative and, importantly, are W -invariant.

Indeed, one has: $\overline{\Pi}_b' = \prod_{i=1}^n (\overline{\Pi}_i')^{n_i}$ for $B_- \ni b = -\sum n_i \omega_i$. Provided that all $n_i > 0$, the *reduced* decomposition $b = b_- = w_0 \pi_{b_+}$ holds for the longest element $w_0 \in W$ and $b_+ = w_0(b) \in B_+$; see (1.18). Thus $\overline{\Pi}_b'$ is divisible on the left by $(\overline{T}_i + 1)$ for any $i > 0$ and therefore divisible by the W -symmetrizer on the left. It results in the W -invariance of $\overline{\Pi}_b'$ provided that $b \in B_-$.

The W -invariance of $\{\overline{\Pi}_b', b \in B_-\}$ simplifies significantly the relation of the \overline{E} -polynomials to the \overline{P} -polynomials. It becomes

$$(2.34) \quad \overline{P}_b = \overline{E}_b \text{ for } b = b_- \in B_-.$$

We come to the following explicit proposition.

Proposition 2.3. *In the representation $\bar{\mathcal{V}}$ of $\bar{\mathcal{H}}^b$,*

$$(2.35) \quad \begin{aligned} \tau_+(\bar{T}'_{\hat{w}})(1) &= q^{-(b,b)/2} \bar{E}_b \text{ for } \hat{w} = \pi_b, b \in B, \\ \bar{\Pi}'_b(1) &= q^{(b,b)/2} \bar{P}_b = q^{(b,b)/2} \bar{E}_b \text{ for } b \in B_-, \end{aligned}$$

where $\bar{\Pi}'_i$ can be replaced by their restrictions $\text{Red}_W(\bar{\Pi}'_i)$ to $\bar{\mathcal{V}}^W$, which are pairwise commutative W -invariant difference operators. \square

Define $\Sigma(b)$ (resp. $\Sigma_*(b), \Sigma_+(b)$) to be the span of those monomials X_c with $c \in \sigma(b)$ (resp. $\sigma_*(b), \sigma_+(b)$); cf. (1.20). Recall the partial ordering \succeq from (1.21).

Proposition 2.4. *For any $b \in B$, one has*

$$(2.36) \quad \bar{E}_b = \sum_{c \succeq b, c \in W(b)} X_c \pmod{\Sigma_+(b)}.$$

Proof. We argue by induction on $l(w_b)$; recall that w_b is, by definition, the unique element of W of shortest length such that $w_b(b_+) = b$. For $b = b_+$, (2.36) clearly holds. Before the inductive step, we remark that the intertwiners \bar{T}'_i ($i > 0$) preserve $\Sigma_+(b)$. Modulo $\Sigma_+(b)$, one has for $i > 0$ that

$$(2.37) \quad \bar{T}'_i(X_b) = \begin{cases} X_{s_i(b)} + X_b, & \text{if } (b, \alpha_i) > 0, \\ X_b, & \text{if } (b, \alpha_i) = 0, \\ 0, & \text{if } (b, \alpha_i) < 0. \end{cases}$$

Now suppose $l(w_b) > 1$ and choose any $i > 0$ such that $l(s_i w_b) < l(w_b)$. Then $w_b = s_i w_c$ is reduced for $c = s_i(b) = s_i w_b(b_+) \succeq b$. One has $(c, \alpha_i) > 0$ and hence $\bar{E}_b = \bar{T}'_i(\bar{E}_c)$.

We must show that for any $w \leq w_b$, the monomial $X_{w(b_+)}$ appears in \bar{E}_b with coefficient 1. Now $w \leq w_b$ implies

$$(i) \ w \leq w_c \text{ or } (ii) \ s_i w \leq w_c \text{ (or both)}.$$

Assuming (i), \bar{E}_c contains $X_{w(b_+)}$ with coefficient 1, by induction. Then either $(w(b_+), \alpha_i) \geq 0$, in which case (2.37) applies directly, or $(w(b_+), \alpha_i) < 0$. In the latter case, $s_i w < w \leq w_c$ and one applies (2.37) to $X_{s_i w(b_+)}$.

Case (ii) can be handled by a similar argument. \square

Comment. For the affine root systems considered in this paper (with α_0 defined in terms of the maximal *short* root ϑ), a connection was established between the polynomials \overline{E}_b and the level-one Demazure characters of the corresponding irreducible affine Lie algebras; see [San] and, especially, Theorem 1 from [Ion]. Paper [Ion] is based on the technique of intertwiners (from [KS] in the GL_n -case and [C6] for arbitrary reduced root systems).

It is important that only positive powers of q appear in the coefficients of \overline{E}_b (see the discussion following (2.30) above). In fact, the coefficients of these q -polynomials are non-negative. One can obtain it from the interpretation via Demazure characters or using the intertwiners (we are going to discuss the latter in further papers). As $q \rightarrow 0$, the polynomials \overline{P}_{b_-} become the classical finite dimensional Lie characters, which can be seen, for instance, from (2.44) below. \square

Concerning the orthogonality of the \overline{E} -polynomials, the μ -function from (2.38) becomes

$$(2.38) \quad \overline{\mu} \stackrel{\text{def}}{=} \mu(t_\nu = 0) = \prod_{\alpha \in R_+} \prod_{j=0}^{\infty} (1 - X_\alpha q_\alpha^j)(1 - X_\alpha^{-1} q_\alpha^{j+1}).$$

The constant term formula becomes a well known identity:

$$(2.39) \quad \langle \overline{\mu} \rangle = \prod_{i=1}^n \prod_{j=1}^{\infty} \frac{1}{1 - q_i^j}.$$

The polynomials \overline{E}_b can be uniquely determined from the relations (2.13) as $t_\nu = 0$:

$$(2.40) \quad \overline{E}_b - X_b \in \oplus_{c \succ b} \mathbb{Q}'_q X_c, \quad \langle \overline{E}_b X_c^{-1} \overline{\mu} \rangle = 0 \quad \text{for } B \ni c \succ b.$$

To state the counterpart of the norm formula (2.18) as $t_\nu = 0$ we will need the limits \overline{E}_b^\dagger of the E -polynomials as $t_\nu \rightarrow \infty$. More generally, we set $\overline{f}^\dagger \stackrel{\text{def}}{=} \lim_{t_\nu \rightarrow \infty} f$ for any Laurent polynomial or series depending on q, t_ν , provided the existence of this limit. Using the conjugation \star from (2.12) (sending $t_\nu^{1/2}$ to $t_\nu^{-1/2}$),

$$(2.41) \quad \overline{(f^\star)} = (\overline{f}^\dagger)^\star, \quad \overline{(f^\star)}^\dagger = (\overline{f})^\star,$$

where $X_b^\star \stackrel{\text{def}}{=} X_b^{-1}$, $(q^{1/(2m)})^\star \stackrel{\text{def}}{=} q^{-1/(2m)}$.

Using this notation, the limit of the norm formula from (2.18) as $t \rightarrow 0$ is as follows:

$$(2.42) \quad \begin{aligned} \langle \overline{E}_b, \overline{E}_c \rangle_\circ &\stackrel{\text{def}}{=} \lim_{t \rightarrow 0} \langle E_b, E_c \rangle_\circ \\ &= \langle \overline{E}_b (\overline{E}_c^\dagger)^* \overline{\mu}_\circ \rangle_\circ = \delta_{bc} \prod_{[\alpha, j]} (1 - q_\alpha^j), \end{aligned}$$

where the last product runs over all $[\alpha, j] \in \lambda'(\pi_b)$ with simple $\alpha = \alpha_i \in R_+$. Use that $(\rho_k, \alpha) = (\rho_k, \nu_\alpha \alpha^\vee)$ and $(\rho_k, \alpha_i^\vee) = k_i$; recall that $q_\alpha = q^{\nu_\alpha}$ and $t_\alpha = q_\alpha^{k_\alpha}$.

Assuming that q is not a root of unity, these formulas readily provide the existence of the polynomials \overline{E}_b^\dagger for any $b \in B$ and that they form a basis of $\overline{\mathcal{V}}$. This holds in fact for any nonzero q , but the justification requires a different approach. For instance, we use (2.30) in the limit $t_\nu \rightarrow \infty$ to establish Corollary 2.6 below.

The relation between the limits $t_\nu \rightarrow 0$ and $t_\nu \rightarrow \infty$ goes through the general formula

$$(2.43) \quad E_b^\star = \prod_{\nu \in \nu_R} t_\nu^{l_\nu(u_b) - l_\nu(w_0)/2} T_{w_0}(E_{\varsigma(b)}), \quad \text{where } \varsigma(b) = -w_0(b),$$

from [C8] and other first author's works. This connection becomes especially simple for the symmetric polynomials:

$$P_b(X)^\star = P_b(X^{-1}) \quad \text{for } b = b_-, \quad \overline{P}_b = P_b(t_\nu \rightarrow 0) = P_{\varsigma(b)}(t_\nu \rightarrow \infty).$$

We use that $P_b(X^{-1}) = P_{\varsigma(b)}(X)$. This formula readily follows from the relations $\widehat{w}(\mu)/\mu = (\widehat{w}(\mu)/\mu)^\star$ for any $\widehat{w} \in \widehat{W}$ if one uses the definition of $\{P_b\}$ via μ . The ratios $\widehat{w}(\mu)/\mu$ are rational functions in terms of $X_{\widehat{\alpha}}, q, t_\alpha$, so the conjugation (applying \star) is well defined.

For $b, c \in B_-$, the norm formula from (2.24) reads as:

$$(2.44) \quad \langle \overline{P}_b(X) \overline{P}_c(X^{-1}) \overline{\mu}_\circ \rangle = \delta_{bc} \prod_{i=1}^n \prod_{j=1}^{-(\alpha_i^\vee, b)} (1 - q_i^j).$$

We use that $P_b^\star = P_{\varsigma(b)} = P_b(X^{-1})$ for $b \in B_-$, which makes defining \overline{P}_b^\dagger unnecessary in this case.

2.5. The limit $t \rightarrow \infty$. Let us discuss the limits \overline{E}_b^\dagger of the E -polynomials more systematically. As orthogonal polynomials, they can be introduced using (2.42); let us outline an approach based on the real integration instead of taking the constant term.

We will use that

$$\mu^\dagger \stackrel{\text{def}}{=} \mu(X; q, t^{-1})^{-1} = \prod_{\tilde{\alpha} \in \tilde{R}_+} \frac{1 - t_\alpha^{-1} X_{\tilde{\alpha}}}{1 - X_{\tilde{\alpha}}}$$

satisfies the relations $\widehat{w}^{-1}(\mu)/\mu = \widehat{w}^{-1}(\mu^\dagger)/\mu^\dagger$ for $\widehat{w} \in \widehat{W}$. Thus μ/μ^\dagger is (formally) \widehat{W} -invariant and either function can be used for the corresponding orthogonal polynomials and operators depending on the setting of the theory. See formula (2.7) from Part I of [CM]. Recall that $X_{\tilde{\alpha}} = q_\alpha^j X_\alpha$ for $\tilde{\alpha} = [\alpha, \nu_\alpha j] \in \tilde{R}$.

In the limit $t_\nu \rightarrow \infty$,

$$(2.45) \quad \overline{\mu}^\dagger = \prod_{\alpha \in R_+} \prod_{j=0}^{\infty} \frac{1}{(1 - X_\alpha q_\alpha^j)(1 - X_\alpha^{-1} q_\alpha^{j+1})}.$$

We need to replace the constant term functional $\langle f \rangle$ by

$$\int_\epsilon f(X) \stackrel{\text{def}}{=} \sum_{w \in W} \int_{w(\iota\epsilon + \mathbb{R}^n)} f(q^x) dx \quad \text{for } \epsilon \in \mathbb{R}^n,$$

provided that $(\epsilon, \alpha) > 0$ for $\alpha \in R_+$. We set $X = q^x$, $X_b = q^{(x,b)}$, and

$$\int_{\iota\epsilon + \mathbb{R}^n} (\cdot) dx = \int_{\iota\epsilon_n - \infty}^{\iota\epsilon_n + \infty} \cdots \int_{\iota\epsilon_1 - \infty}^{\iota\epsilon_1 + \infty} (\cdot) dx_1 \cdots dx_n,$$

where $x_i = (x, \omega_i)$; the integration contours $w(\iota\epsilon + \mathbb{R}^n)$ are the images of $\iota\epsilon + \mathbb{R}^n$. The integral $\int_\epsilon \overline{\mu}^\dagger$ is connected with the Appell-Lerch sums and will not be discussed here; it does not depend on the choice of ϵ . Cf. Section 2.3.5, “Etingof’s theorem”, from [C8].

Now $\{\overline{E}_b^\dagger\}$ can be introduced by means of the relations

$$(2.46) \quad \overline{E}_b^\dagger - X_b \in \oplus_{c \succ b} \mathbb{Q}'_q X_c, \quad \int_\epsilon \overline{E}_b^\dagger X_c^{-1} \overline{\mu}^\dagger = 0 \quad \text{for } B \ni c \succ b.$$

Setting $\overline{\mu}_\epsilon^\dagger = \overline{\mu}^\dagger / \int_\epsilon \overline{\mu}^\dagger$, the norm formula (2.18) becomes

$$(2.47) \quad \langle \overline{E}_b^\dagger, \overline{E}_c^\dagger \rangle_\epsilon \stackrel{\text{def}}{=} \int_\epsilon \overline{E}_b^\dagger (\overline{E}_c)^\ast \overline{\mu}_\epsilon^\dagger = \delta_{bc} \prod_{[\alpha, j]} (1 - q_\alpha^{-j}),$$

where $[\alpha, j] \in \lambda'(\pi_b)$ for simple α .

Calculating the integrals here can be reduced to taking the constant term. Namely,

$$\int_{\epsilon} \overline{E}_b^{\dagger} (\overline{E}_c)^* \overline{\mu}_{\epsilon}^{\dagger} = \langle \overline{E}_b^{\dagger} (\overline{E}_c)^* \overline{\mu}_{\circ}^* \rangle \text{ for } b, c \in B.$$

This follows from the relation $\overline{\mu} \overline{\mu}^{\dagger} = 1$, which connects the action of \widehat{W} on $\overline{\mu}_{\circ}$ and $\overline{\mu}_{\epsilon}^{\dagger}$. Formally, $\overline{\mu}_{\circ}^* / \overline{\mu}^{\dagger}$ is a \widehat{W} -invariant function. We conclude that (2.47) and (2.42) result in coinciding families of polynomials.

Proposition 2.5 below uses the intertwining operators to establish the existence of $\{\overline{E}_b^{\dagger}\}$ in a more direct way; it also relates them to the \overline{E} -polynomials when $b \in B_-$. In Corollary 2.6, we show that the coefficients of $\{\overline{E}_b^{\dagger}\}$ belong to $\mathbb{Z}[q^{-1}]$.

We set

$$\begin{aligned} \ddot{T}_i^{\dagger} &\stackrel{\text{def}}{=} t_i^{-1/2} T_i, & (\ddot{T}_i^{\dagger})' &\stackrel{\text{def}}{=} t_i^{-1/2} T_i^{-1}, \\ \overline{T}_i^{\dagger} &\stackrel{\text{def}}{=} \ddot{T}_i^{\dagger}(t_i = \infty), & (\overline{T}_i^{\dagger})' &\stackrel{\text{def}}{=} (\ddot{T}_i^{\dagger})'(t_i = \infty) = \overline{T}_i^{\dagger} - 1, \\ \overline{T}_i^{\dagger} &= (s_i + 1) \frac{1}{1 - X_{\alpha_i}}, & (\overline{T}_i^{\dagger})' &= \frac{X_{\alpha_i}}{X_{\alpha_i} - 1} (s_i - 1) \text{ in } \overline{\mathcal{V}}. \end{aligned}$$

Recall that $\overline{\mathcal{V}} = \mathbb{Q}_q[X_b, b \in B]$ as a space; by $\overline{\mathcal{V}}^{\dagger}$, we will mean this space with the action of \overline{T}_i^{\dagger} and other \dagger -operators.

Correspondingly, we set $\ddot{Y}_a^{\dagger} = q^{-(a_+, \rho_k)} Y_a$. Then it is straightforward to see that the limit

$$(2.48) \quad \overline{Y}_a^{\dagger} \stackrel{\text{def}}{=} \lim_{t \rightarrow 0} \ddot{Y}_a^{\dagger}$$

exists. Using (2.15), we arrive at the analog of (2.33):

$$(2.49) \quad \overline{Y}_a^{\dagger}(\overline{E}_b^{\dagger}) = \begin{cases} q^{-(a,b)} \overline{E}_b^{\dagger} & \text{if } u_b(a) = a_+, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5. (i) For $b \in B_-$,

$$(2.50) \quad \overline{E}_b^{\dagger} = q^{(\rho, b)} (\overline{T}'_{\pi_{\rho}}(\overline{E}_{-w_0(b)}))^*,$$

where $\overline{T}'_{\pi_{\rho}} = \overline{T}'_{j_1} \cdots \overline{T}'_{j_l} \pi_r^{-1}$ is defined for a reduced decomposition $\pi_{\rho} = \pi_r s_{j_l} \cdots s_{j_1}$ ($r \in O$) and does not depend on the choice of such a decomposition.

(ii) In the opposite direction, the polynomials $\{\overline{E}_b^*\}$ for $b \in B_-$ can be obtained from $\overline{E}_{-w_0(b)}^\dagger$ as follows:

$$(2.51) \quad \overline{E}_b^* = \overline{T}_{w_0}^\dagger(\overline{E}_{-w_0(b)}^\dagger).$$

In particular, the polynomials on the right-hand side are W -invariant and nonzero for any $b \in B_-$ with coefficients in $\mathbb{Z}_+[q^{-1}]$.

(iii) If $(b, \alpha_i) < 0$ ($1 \leq i \leq n$), then

$$(2.52) \quad \overline{E}_{s_i(b)}^\dagger = \begin{cases} (1 - q^{(b, \alpha_i)})^{-1} (\overline{T}_i^\dagger)'(\overline{E}_b^\dagger) & \text{if } (u_b(\alpha_i), \rho^\vee) = 1, \\ (\overline{T}_i^\dagger)'(\overline{E}_b^\dagger) & \text{if } (u_b(\alpha_i), \rho^\vee) > 1. \end{cases}$$

Combining this with (2.50), we obtain another proof of the existence of \overline{E}_b^\dagger for any $b \in B$, which holds for any $q \neq 0$ due to (2.30) (see Corollary 2.6 below).

(iv) Assuming that $(b, \alpha_i) > 0$ ($1 \leq i \leq n$), we set $\overline{T}_{i,b}^\dagger = \lim_{t_\nu \rightarrow \infty} \ddot{T}_{i,b}^\dagger$ for $\ddot{T}_{i,b}^\dagger = q^{-(b, \alpha_i)} t_i^{1/2} T_i Y_b^{-1}$. Then

$$(2.53) \quad \overline{E}_{s_i(b)}^\dagger = \begin{cases} (\overline{T}_{i,b}^\dagger + q^{-(b, \alpha_i)})(\overline{E}_b^\dagger) & \text{if } (u_b(\alpha_i), \rho^\vee) = -1, \\ \overline{T}_{i,b}^\dagger(\overline{E}_b^\dagger) & \text{if } (u_b(\alpha_i), \rho^\vee) < -1. \end{cases}$$

Proof. Claim (i). Let $b = b_-$. We will normalize (2.43) to prove (2.50). Note that $q^{(c, w_0(b) + \rho_k)} Y_c^{-1}$ acts as the identity on $E_{-w_0(b)}$ for any $c \in B$. Taking $c = c_+$, so that $l_\nu(c) = 2(c, \rho_\nu^\vee)$, one therefore has

$$(2.54) \quad E_b^* = q^{(c, w_0(b))} \prod_{\nu} t_\nu^{-l_\nu(w_0)/2 + l_\nu(c)/2} T_{w_0} Y_c^{-1}(E_{-w_0(b)}).$$

Specializing further to $c = \rho$, we have $Y_\rho = T_{\pi_\rho} T_{w_0}$ and (2.54) becomes

$$(2.55) \quad E_b^* = q^{-(\rho, b)} \prod_{\nu} t_\nu^{l_\nu(\pi_\rho)/2} T_{\pi_\rho}^{-1}(E_{-w_0(b)}) = q^{-(\rho, b)} \ddot{T}'_{\pi_\rho}(E_{-w_0(b)}),$$

where by definition $\ddot{T}'_{\pi_\rho} = \ddot{T}'_{j_1} \cdots \ddot{T}'_{j_l} \pi_r^{-1}$ for any reduced decomposition $\pi_\rho = \pi_r s_{j_l} \cdots s_{j_1}$. Moving \star to the right-hand side and taking $t_\nu \rightarrow \infty$, we obtain (2.50).

Claim (ii). This is immediate from (2.43), which is (2.54) for $c = 0$. Note that (2.51) modulo $\Sigma_+(b)$ results in (2.36) for $b = b_-$.

Claim (iii). This follows from a modification of (2.25) and (2.26). It is convenient to use the normalized intertwiners $\mathfrak{G}_i \stackrel{\text{def}}{=} \psi_i^{-1} \Psi_i$ for

$$(2.56) \quad \Psi_i = \tau_+(T_i) + \frac{t_i^{1/2} - t_i^{-1/2}}{Y_{\alpha_i}^{-1} - 1}, \quad \psi_i = t_i^{1/2} + \frac{t_i^{1/2} - t_i^{-1/2}}{Y_{\alpha_i}^{-1} - 1}.$$

For simplicity, let us take here $1 \leq i \leq n$. In addition to the braid relations, the normalized intertwiners satisfy $\mathfrak{G}_i^2 = 1$. Hence $\Psi_i^{-1} = \psi_i^{-1} \Psi_i \psi_i^{-1}$. Now, when $(b, \alpha_i) < 0$, (2.26) gives $E_{s_i(b)} = t_i^{-1/2} \Psi_i^{-1}(E_b)$. Applying $\Psi_i^{-1} = \psi_i^{-1} \Psi_i \psi_i^{-1}$ to E_b , the first ψ_i^{-1} produces

$$\left(t_i^{1/2} + \frac{t_i^{1/2} - t_i^{-1/2}}{q^{(\alpha_i, b_\#)} - 1} \right)^{-1} = t_i^{1/2} \frac{q^{(\alpha_i, b_\#)} - 1}{q^{(\alpha_i, b_\#)} t_i - 1}.$$

Moving ψ_i^{-1} through Ψ_i changes $Y_{\alpha_i}^{-1}$ to Y_{α_i} . Thus, the second (left) ψ_i^{-1} produces the factor

$$\left(t_i^{1/2} + \frac{t_i^{1/2} - t_i^{-1/2}}{q^{-(\alpha_i, b_\#)} - 1} \right)^{-1} = t_i^{-1/2} \frac{q^{-(\alpha_i, b_\#)} - 1}{q^{-(\alpha_i, b_\#)} - t_i^{-1}} = t_i^{-1/2} \frac{q^{(\alpha_i, b_\#)} - 1}{q^{(\alpha_i, b_\#)} t_i^{-1} - 1}.$$

Multiplying these two factors and taking $t_\nu \rightarrow \infty$, one arrives at (2.52). Note that $u_b(\alpha_i) > 0$ and hence $q^{(\alpha_i, b_\#)}$ contains non-positive powers of t_ν and at least one t_i^{-1} .

Formula (2.52) can be directly deduced from (2.26) and the norm formula (2.18) in the limit $t \rightarrow \infty$. Indeed, for $i > 0$ and due to the inequality $(b, \alpha_i) < 0$,

$$(2.57) \quad \frac{E_c}{\langle E_c, E_c \rangle_\circ} = t_i^{-\frac{1}{2}} \Psi_i^b \left(\frac{E_b}{\langle E_b, E_b \rangle_\circ} \right).$$

Then $\lim_{t \rightarrow \infty} t_i^{-\frac{1}{2}} \Psi_i^b = \overline{T}_i^\dagger - 1 = (\overline{T}_i^\dagger)'$ and

$$\lim_{t \rightarrow \infty} \langle E_b, E_b \rangle_\circ = \prod_{[\alpha, j]} (1 - q_\alpha^{-j}) = (1 - q^{(b, \alpha_i)}) \lim_{t \rightarrow \infty} \langle E_c, E_c \rangle_\circ.$$

The product here is over $[\alpha, j] \in \lambda'(\pi_b)$ for simple α . Cf. (2.47) and recall that $q_\alpha = q^{\nu_\alpha} = q^{2/(\alpha, \alpha)}$.

Claim (iv). Using (2.26) and (2.15), we can write

$$(2.58) \quad E_{s_i(b)} = t_i^{1/2} \left(T_i + \frac{t_i^{1/2} - t_i^{-1/2}}{q^{(\alpha_i, b_\#)} - 1} \right) q^{-(b, b_\#)} Y_b^{-1}(E_b).$$

We claim that $t_i^{1/2} q^{-(b, b_\#)} T_i Y_b^{-1}$ is well defined as $t_\nu = \infty$. Indeed,

$$l_\nu(b) = 2(b_+, \rho_\nu^\vee) = -2(b_-, \rho_\nu^\vee)$$

and since $(b, \alpha_i) > 0$, u_b has a reduced decomposition ending in s_i . Hence (1.42) gives the claim.

Now taking $t_\nu \rightarrow \infty$ in (2.58), while noting that

$$(2.59) \quad \lim_{t_\nu \rightarrow \infty} \frac{t_i - 1}{q^{(\alpha_i, b_\#)} - 1} = \begin{cases} q^{-(\alpha_i, b)}, & \text{if } (u_b(\alpha_i), \rho^\vee) = -1, \\ 1, & \text{if } (u_b(\alpha_i), \rho^\vee) < -1. \end{cases}$$

we arrive at (2.53). \square

Corollary 2.6. (i) For $b \in B_-$, one has

$$(2.60) \quad \overline{E}_b^\dagger = X_b + \sum_{W(b) \ni c \twoheadrightarrow b} q^{n_b(c)} X_c \pmod{\Sigma_+(b)}, \text{ where } n_b(c) \in \mathbb{Z}_-.$$

(ii) The coefficients of \overline{E}_b^\dagger belong to $\mathbb{Z}[q^{-1}]$ for any $b \in B$.

Proof. (i) The proof is straightforward using (2.50) and (2.36).

(ii) By (2.30), the denominators of the coefficients in E_b are of products of factors of the form

$$(1 - q^j \prod_\nu t_\nu^{m_\nu}), \text{ where } j, \sum_\nu m_\nu > 0.$$

By Proposition 2.5(iii), we already know that the \overline{E}_b^\dagger exist, at least with coefficients in \mathbb{Q}'_q . Hence we may set $t = t_\nu$ for all ν when calculating the limits of the coefficients. As polynomials in t , the denominators of E_b then have leading terms of the form $\pm q^r t^s$ where $r, s > 0$. Since \overline{E}_b^\dagger exists, no higher power of t can appear in the corresponding numerator. Therefore, we see that the coefficients of \overline{E}_b^\dagger belong to $\mathbb{Z}[q^{\pm 1}]$. Using (i) and then (2.52) inductively, now it is easy to see that the coefficients of \overline{E}_b^\dagger lie in $\mathbb{Z}[q^{-1}]$. \square

Positivity conjecture. We conjecture that the coefficients of \overline{E}_b^\dagger for all $b \in B$ belong to $\mathbb{Z}_+[q^{-1}]$, which will hopefully follow from a more systematic theory of the intertwiners in the nil-case.

For $b \in B_-$, we expect that the polynomials \overline{E}_b^\dagger and their untwisted counterparts (not considered in this work) coincide with the corresponding level-one Demazure characters for the Kac-Moody algebra $\widehat{\mathfrak{g}}$ associated with \widetilde{R} upon shifting the q -powers by the PBW-degrees

from [FFL] (twisted or untwisted). We thank Evgeny Feigin for his help with settling the conjecture below, which is directly related to our ongoing joint research project with him.

It would give the positivity of the coefficients of \overline{E}_b^\dagger for $b \in B_-$. Moreover, then (2.52) can be generally applied to verify that the coefficients of \overline{E}_b^\dagger are from $\mathbb{Z}_+[q^{-1}]$ for *all* $b \in B$, though their “geometric” meaning is unclear to us. We suspect here a connection with the *local Weyl modules* considered under the PBW-filtration.

Since the relations of \overline{E}_b to the level-one Demazure characters holds only in the twisted case [Ion] and simply to avoid giving the definitions of the untwisted dag-polynomials and the twisted PBW-filtration, we state the conjecture in this paper only for $\widehat{\mathfrak{g}}$ of ADE-type. Let $\widehat{\mathfrak{b}}_+ \supset \widehat{\mathfrak{n}}_+$ be the Borel subalgebra and its radical.

Conjecture 2.7. *For $b \in B_-$, let \mathcal{V}_{-b} be the Demazure module in the so-called basic representation of $\widehat{\mathfrak{g}}$, which is a $\widehat{\mathfrak{b}}_+$ -module generated by the extremal vector v_{-b} of weight $-b \in B_+$. In the setting from [San, Ion],*

$$\overline{E}_b = \sum_{c \succ b, g} \dim_{-c, g} q^g X_c \text{ for } c \in B,$$

where $\dim_{-c, g} = \dim_{\mathbb{C}} V_{-c, g}$ for the subspace $V_{-c, g}$ of the vectors of degree $g \in \mathbb{Z}_+$ in the subspace $V_{-c} \subset \mathcal{V}_{-b}$ of weight $-c$ for $c \in B$ and the standard Kac-Moody grading. Then

$$\overline{E}_b^\dagger = \sum_{B \ni c \succ b, f \geq 0} \dim_{\mathbb{C}} (\mathcal{G}_f \cap V_{-c, g} / \mathcal{G}_{f-1} \cap V_{-c, g}) q^{-f-g} X_c, \text{ where}$$

$$\mathcal{G}_{-1} = \{0\}, \mathcal{G}_0 = \mathbb{C}v_{-b}, \mathcal{G}_f = \widehat{\mathfrak{n}}_+(\mathcal{G}_{f-1}) + \mathcal{G}_{f-1} \text{ for } f > 0.$$

In particular, $-n_b(w(b))$ for $w \in W$ defined in (2.60) equals the minimal number of $\beta \in R_+$ such that $w(b) - b = \sum \beta$ in the case of A_n .

□

Connection maps. Let us introduce $\dot{\mathcal{H}}^{b, \dagger}$ as the subalgebra of \mathcal{H}^b generated over $\ddot{\mathbb{Q}}_{q, t}^\dagger$ by the elements

$$(2.61) \quad X_b \ (b \in B), \ \ddot{T}_i^\dagger \ (i \geq 0), \ \Pi^b.$$

Cf. Definition 1.6. The defining relations of $\dot{\mathcal{H}}^{b, \dagger}$ in terms of these generators are:

- (o) $(\ddot{T}_i^\dagger - 1)(\ddot{T}_i^\dagger + t_i^{-1}) = 0, 0 \leq i \leq n;$
- (i) $\ddot{T}_i^\dagger \ddot{T}_j^\dagger \ddot{T}_i^\dagger \cdots = \ddot{T}_j^\dagger \ddot{T}_i^\dagger \ddot{T}_j^\dagger \cdots, m_{ij} \text{ factors on each side};$
- (ii) $\pi_r \ddot{T}_i^\dagger \pi_r^{-1} = \ddot{T}_j^\dagger \text{ if } \pi_r(\alpha_i) = \alpha_j, \pi_r \in \Pi^\flat;$
- (iii) $\ddot{T}_i^\dagger X_b = X_b X_{\alpha_i}^{-1} (\ddot{T}_i^\dagger)'$ if $(b, \alpha_i^\vee) = 1, 0 \leq i \leq n;$
- (iv) $\ddot{T}_i^\dagger X_b = X_b \ddot{T}_i^\dagger$ if $(b, \alpha_i^\vee) = 0$ for $0 \leq i \leq n;$
- (v) $\pi_r X_b \pi_r^{-1} = X_{\pi_r(b)} = X_{u_r^{-1}(b)} q^{(\omega_{r^*}, b)}.$

Let $\overline{\mathcal{H}}^{b, \dagger}$ be the specialization of $\dot{\mathcal{H}}^{b, \dagger}$ as all $t_\nu = \infty$ (i.e., $t_\nu^{-1} = 0$), and $\overline{\mathcal{V}}^\dagger$ the image of the polynomial representation under this specialization. The involution η from (1.53) gives the following isomorphisms:

$$(2.62) \quad \eta : \overline{\mathcal{H}}^b \rightarrow \overline{\mathcal{H}}^{b, \dagger}$$

$$\overline{T}_i \mapsto (\overline{T}_i^\dagger)', \quad X_b \mapsto X_b^{-1}, \quad \pi_r \mapsto \pi_r, \quad q \mapsto q^{-1}$$

$$(2.63) \quad \eta^\dagger \stackrel{\text{def}}{=} \eta^{-1} : \overline{\mathcal{H}}^{b, \dagger} \mapsto \overline{\mathcal{H}}^b$$

$$\overline{T}_i^\dagger \mapsto \overline{T}_i', \quad X_b \mapsto X_b^{-1}, \quad \pi_r \mapsto \pi_r, \quad q \mapsto q^{-1}.$$

Due to (2.19), one has

$$(2.64) \quad (H(f))^* = \eta(H)(f^*), \quad \text{for } f \in \overline{\mathcal{V}}, \quad H \in \overline{\mathcal{H}}^b$$

$$(2.65) \quad (H(f))^* = \eta^\dagger(H)(f^*), \quad \text{for } f \in \overline{\mathcal{V}}^\dagger, \quad H \in \overline{\mathcal{H}}^{b, \dagger}.$$

3. NONSYMMETRIC WHITTAKER FUNCTION

Let us recall the definition of the Ruijsenaars-Etingof limiting procedure from [Ru, Et] employed and developed in [C10] (the symmetric theory). Given a difference operator \mathcal{L} and a function F , it is defined by

$$(3.1) \quad \mathfrak{ae}(\mathcal{L}) = (X_{\rho_k} \Gamma_{-\rho_k}) \mathcal{L} (X_{\rho_k} \Gamma_{-\rho_k})^{-1}, \quad \mathfrak{ae}(F) = X_{\rho_k} \Gamma_{-\rho_k}(F)$$

$$RE(A) = \lim_{k \rightarrow \infty} \mathfrak{ae}(A), \quad RE(F) = \lim_{k \rightarrow \infty} \mathfrak{ae}(F).$$

This procedure was applied in [C10] to obtain global Whittaker functions from global *symmetric* q, t -spherical functions in the symmetric case. The existence a nonsymmetric analogue of this procedure remained an entirely open question until [CM], where it was shown for the root system A_1 as an application of W -spinors. In [CO2], a systematic algebraic study the corresponding nonsymmetric (spinor) Whittaker

functions (still for A_1 only) was carried out; this involved a detailed analysis of certain subalgebras of the nil-DAHA, most importantly the *core subalgebra*.

In this section, we develop a nonsymmetric (spinor) variant of the Ruijsenaars-Etingof procedure and apply it to global *nonsymmetric* q, t -spherical function. Algebraically, these constructions are closely related with the theory of *pseudo-polynomial representation* of nil-DAHA, which is introduced at the end of this section and will be the subject of our future works.

3.1. Global spherical functions. By the *Gaussians* we mean

$$(3.2) \quad \tilde{\gamma}^{\oplus} = \sum_{b \in B} q^{-(b,b)/2} X_b, \quad \tilde{\gamma}^{\ominus} = \sum_{b \in B} q^{(b,b)/2} X_b.$$

We need mainly the Gaussian $\tilde{\gamma}^{\ominus}$ in this paper. The multiplication by $\tilde{\gamma}^{\ominus}$ preserves the space of Laurent series with coefficients in $\mathbb{Q}[t][[q^{\frac{1}{2\tilde{m}}}]$, where \tilde{m} is from the definition of $\mathbb{Q}'_{q,t}$.

The Gaussians satisfies the fundamental difference equations

$$(3.3) \quad \Gamma_a(\tilde{\gamma}^{\oplus}) = q^{(a,a)/2} X_a \tilde{\gamma}^{\oplus}, \quad \Gamma_a(\tilde{\gamma}^{\ominus}) = q^{-(a,a)/2} X_a^{-1} \tilde{\gamma}^{\ominus} \quad \text{for } a \in B.$$

Later we will need the following special case of (3.3):

$$(3.4) \quad \Gamma_{\rho_k}(\tilde{\gamma}^{\ominus}) = q^{-(\rho_k, \rho_k)/2} X_{-\rho_k} \tilde{\gamma}^{\ominus},$$

provided $\rho_k \in B$ (when $B = P$, the condition $k_{\nu} \in \mathbb{Z}$ is sufficient).

If one uses here the *real Gaussians* defined as

$$(3.5) \quad \gamma^{\pm 1} = q^{\pm x^2/2}, \quad \text{where } X_b \stackrel{\text{def}}{=} q^{x_b}, x_b = (x, b), x^2 = \sum_i x_{\alpha_i} x_{\omega_i^{\vee}},$$

then (3.3) is satisfied for any complex a . Note that if the series $\tilde{\gamma}^{\ominus}$ is considered as a holomorphic function for $|q| < 1$, then the function $\tilde{\gamma}^{\ominus} \gamma$ is B -periodic in terms of x .

We assume that $|q| < 1$ and use the notation $\tilde{\gamma}_{\lambda}^{\ominus}$ for the Gaussians defined for the variable $\Lambda = q^{\lambda}$. Let $\tilde{\gamma}_x^{\ominus} = \tilde{\gamma}^{\ominus}$ for the sake of uniformity. Thus, $\tilde{\gamma}_{\lambda}^{\ominus} = \tilde{\gamma}^{\ominus}(q^{\lambda})$. Accordingly, we use superscripts when applying operators from the polynomial representation of \mathcal{H} to functions of X or Λ . For instance, we write T_i^{λ} for the action of T_i from (2.1) on functions of $\Lambda = q^{\lambda}$, where we replace X_{α_i} by Λ_{α_i} . When no superscript is used, the action is understood in terms of X .

We will also use the normalization constant

$$(3.6) \quad \tilde{\gamma}^\ominus(q^{\rho_k}) = \sum_{a \in B} q^{\frac{(a,a)}{2} + (\rho_k, a)}.$$

The following theorem results from Theorem 5.4 and Corollary 7.3 of [C5]. The function $G(X, \Lambda)$ introduced in (3.8) is called *global non-symmetric q, t -spherical function*.

Theorem 3.1. (i) *The Laurent series*

$$(3.7) \quad \Xi(X, \Lambda; q, t) \stackrel{\text{def}}{=} \sum_{b \in B} q^{(b_\sharp, b_\sharp)/2 - (\rho_k, \rho_k)/2} \frac{E_b^*(X) E_b(\Lambda)}{\langle E_b, E_b \rangle_\circ}$$

is well defined with coefficients in $\mathbb{Q}[t][[q^{\frac{1}{2m}}]]$. For $|q| < 1$, Ξ converges to an entire function of X, Λ , provided t_ν are chosen so that all E -polynomials exist (by (2.30), the conditions $|t_\nu| < 1$ are sufficient). Accordingly, $G(X, \Lambda)$ defined via

$$(3.8) \quad \frac{\tilde{\gamma}_x^\ominus \tilde{\gamma}_\lambda^\ominus}{\tilde{\gamma}^\ominus(q^{\rho_k})} G(X, \Lambda) \stackrel{\text{def}}{=} \Xi(X, \Lambda; q, t)$$

is a meromorphic function of X, Λ , which is analytic apart from the zeros of $\tilde{\gamma}_x^\ominus \tilde{\gamma}_\lambda^\ominus$.

(ii) *The function $G(X, \Lambda)$ satisfies*

$$(3.9) \quad G(X, \Lambda) = G(\Lambda, X), \quad T_i^x(G(X, \Lambda)) = T_i^\lambda(G(X, \Lambda)), \quad 1 \leq i \leq n,$$

and the following extension of (2.15):

$$(3.10) \quad Y_a(G(X, \Lambda)) = \Lambda_a^{-1} G(X, \Lambda) \quad \text{for } a \in B.$$

For an arbitrary $b \in B$, one has

$$(3.11) \quad G(X, q^{b_\sharp}) = \frac{E_b(X)}{E_b(q^{-\rho_k})} \prod_{\alpha \in R_+} \prod_{j=1}^{\infty} \left(\frac{1 - q^{(\rho_k, \alpha) + \nu_\alpha j}}{1 - t_\alpha^{-1} q^{(\rho_k, \alpha) + \nu_\alpha j}} \right).$$

□

Note that relations (3.9) and (3.10) can be uniformly presented using the anti-involution φ from (1.43) as follows:

$$(3.12) \quad H^x(G(X, \Lambda)) = (\varphi(H))^\lambda(G(X, \Lambda)) \quad \text{for } H \in \mathcal{H}^b.$$

It reflects the fundamental fact that $G(X, \Lambda)$ represents the Fourier transform of DAHA corresponding to the automorphism σ from (1.50), which satisfies the relation $\varphi \sigma \varphi = \sigma^{-1}$; see [C5].

3.2. Action of intertwiners. Relation (3.12) results in the following formulas for the action of the intertwining operators on E_b^* . Let us first modify the intertwiners Ψ_i and $\tau_+(\pi_r)$ from (2.25) and Theorem 2.2 by applying the automorphism τ_-^{-1} , which preserves the elements Y_b, T_i for any $b \in B, i \geq 0$. Namely, we set $Y_0 = Y_{\alpha_0} = q^{-1}Y_{\vartheta}^{-1}$,

$$(3.13) \quad \begin{aligned} \Psi'_{s_i} &= \Psi'_i = \tau_-^{-1}\tau_+(T_i) + \frac{t_i^{1/2} - t_i^{-1/2}}{Y_{\alpha_i}^{-1} - 1}, \quad 0 \leq i \leq n, \\ \Psi_{\pi_r} &= \tau_-^{-1}\tau_+(\pi_r) \quad \text{for } r \in O, \quad \Psi'_{\widehat{w}} = \Psi'_{\pi_r} \Psi'_{i_l} \cdots \Psi'_{i_1}, \end{aligned}$$

where the decomposition $\widehat{W}^b \ni \widehat{w} = \pi_r s_{i_l} \cdots s_{i_1}$ is reduced and $\Psi'_{\widehat{w}}$ does not depend on its choice. Then $\Psi'_{\widehat{w}}(E_b)$ is proportional to E_c for $c = \widehat{w}((b))$ and any $b \in B$. More precisely, using the action $\tau_-^{-1}(E_b) = q^{b_{\sharp}^2/2 - \rho_k^2/2} E_b$ ($b \in B$) in the polynomial representation from [C8], Proposition 3.3.4,

$$(3.14) \quad \begin{aligned} t_i^{1/2} \Psi'_i(q^{(b_+, \rho_k)} E_b) &= q^{(c_+, \rho_k)} E_c, \quad t_i^{-1/2} \Psi'_i\left(\frac{q^{(c_+, \rho_k)} E_c}{\langle E_c, E_c \rangle_{\circ}}\right) = \frac{q^{(b_+, \rho_k)} E_b}{\langle E_b, E_b \rangle_{\circ}}, \\ \text{for } 0 \leq i \leq n, \quad c &= s_i((b)) \quad \text{provided that } (\alpha_i, b + d) > 0, \\ \Psi'_{\pi_r}(q^{(b_+, \rho_k)} E_b) &= q^{(c_+, \rho_k)} E_c \quad \text{for } c = \pi_r((b)) \quad \text{and } r \in O'. \end{aligned}$$

Second, we set $\widetilde{\Psi}'_{\widehat{w}} = \widetilde{\Psi}'_{\pi_r} \widetilde{\Psi}'_{i_l} \cdots \widetilde{\Psi}'_{i_1}$ for induced decompositions $\widehat{W}^b \ni \widehat{w} = \pi_r s_{i_l} \cdots s_{i_1}$, where $0 \leq i \leq n, r \in O$ and

$$(3.15) \quad \widetilde{\Psi}'_{s_i} = \widetilde{\Psi}'_i = \tau_+^{-1}(T_i) + \frac{t_i^{1/2} - t_i^{-1/2}}{\sigma^{-1}(X_{\alpha_i}) - 1}, \quad \widetilde{\Psi}'_{\pi_r} = \tau_+^{-1}(\pi_r).$$

Note that

$$(3.16) \quad \begin{aligned} \tau_-(\sigma^{-1}(X_b)) &= \sigma^{-1}(\tau_+^{-1}(X_b)) = \sigma^{-1}(X_b), \quad \text{which results in} \\ \tau_-(\widetilde{\Psi}'_{\widehat{w}}) &= \widetilde{\Psi}''_{\widehat{w}} \stackrel{\text{def}}{=} \widetilde{\Psi}''_{\pi_r} \widetilde{\Psi}''_{i_l} \cdots \widetilde{\Psi}''_{i_1} \quad \text{for } \widetilde{\Psi}''_{\pi_r} = \sigma^{-1}(\pi_r), \quad r \in O, \\ \widetilde{\Psi}''_i &= \tau_- \tau_+^{-1} \tau_-(T_i) + \frac{t_i^{1/2} - t_i^{-1/2}}{\sigma^{-1}(X_{\alpha_i}) - 1} = \sigma^{-1}(T_i) + \frac{t_i^{1/2} - t_i^{-1/2}}{\sigma^{-1}(X_{\alpha_i}) - 1}. \end{aligned}$$

Therefore both families here do not depend on particular choices of the reduced decompositions of $\widehat{w} \in \widehat{W}^b$ and intertwine $\{\sigma^{-1}(X_b)\}$:

$$(3.17) \quad \widetilde{\Psi}''_{\widehat{w}} \sigma^{-1}(X_b) (\widetilde{\Psi}''_{\widehat{w}})^{-1} = \sigma^{-1}(X_{\widehat{w}(b)}) = \widetilde{\Psi}'_{\widehat{w}} \sigma^{-1}(X_b) (\widetilde{\Psi}'_{\widehat{w}})^{-1},$$

where $b \in B, \widehat{w} \in \widehat{W}^b$.

Proposition 3.2. For $\Xi(X, \Lambda; q, t)$ from (3.7) and $\widehat{w} \in \widehat{W}^b$,

$$(3.18) \quad (\tau_+^{-1} \varphi \tau_+)(\Psi'_{\widehat{w}}) = \widetilde{\Psi}'_{\widehat{w}^{-1}}, \quad (\Psi'_{\widehat{w}})^\lambda(\Xi(X, \Lambda)) = (\widetilde{\Psi}'_{\widehat{w}^{-1}})^x(\Xi(X, \Lambda)).$$

Accordingly, $\widetilde{\Psi}'_{\widehat{w}}(E_b^*)$ is proportional to E_c for $c = \widehat{w}((b))$ and any $b \in B$. More precisely,

$$(3.19) \quad t_i^{\frac{1}{2}} \widetilde{\Psi}'_i \left(\frac{q^{\frac{(b,b)}{2}} E_b^*}{\langle E_b, E_b \rangle_\circ} \right) = \frac{q^{\frac{(c,c)}{2}} E_c^*}{\langle E_c, E_c \rangle_\circ}, \quad t_i^{-\frac{1}{2}} \widetilde{\Psi}'_i \left(q^{\frac{(c,c)}{2}} E_c^* \right) = q^{\frac{(b,b)}{2}} E_b^*$$

for $0 \leq i \leq n$, $b \in B$, $c = s_i((b))$ when $(\alpha_i, b + d) < 0$,
and $\tau_+^{-1}(T_i)(E_b^*) = t_i^{1/2} E_b^*$ when $(\alpha_i, b + d) = 0$, $i \geq 0$,
 $\widetilde{\Psi}'_{\pi_r} \left(\frac{q^{\frac{(b,b)}{2}} E_b^*}{\langle E_b, E_b \rangle_\circ} \right) = \frac{q^{\frac{(c,c)}{2}} E_c^*}{\langle E_c, E_c \rangle_\circ}$ for $c = \pi_r((b))$, $r \in O'$.

Proof. To check (3.18), we use that $\tau_+(T_i) = T_i$ for $i > 0$ and that

$$(3.20) \quad \begin{aligned} \tau_+(\pi_r) &= q^{-\omega_r^2/2} X_{\omega_r} \pi_r = q^{-\omega_r^2/2} X_{\omega_r} T_{u_r^*} Y_{\omega_r^*}^{-1} \quad (r \in O), \\ \tau_+(T_0) &= X_0^{-1} T_0^{-1} = q^{-1} X_{\vartheta} T_{s_{\vartheta}} Y_{\vartheta}^{-1}, \text{ which result in} \\ \varphi(\tau_+(\pi_r)) &= \tau_+(\pi_{r^*}) \text{ for } r \in O, \quad \varphi(\tau_+(T_0)) = \tau_+(T_0). \end{aligned}$$

Use (1.47) and the relation $\pi_r = \pi_{r^*}^{-1}$. Accordingly,

$$\begin{aligned} \tau_+^{-1} \varphi \tau_+(\tau_+^{-1} \tau_+(\pi_r)) &= \tau_+^{-1} \varphi \tau_+(\tau_+^{-1} \varphi \tau_+(\pi_{r^*})) \\ &= (\tau_+^{-1} \tau_- \tau_+^{-1} \tau_+)(\pi_{r^*}) = (\tau_+^{-1} \tau_-)(\pi_{r^*}) = \tau_+^{-1}(\pi_{r^*}), \\ \text{and } \tau_+^{-1} \varphi \tau_+(\tau_+^{-1} \tau_+(T_0)) &= \dots = \tau_+^{-1}(T_0). \end{aligned}$$

Also,

$$\tau_+^{-1} \varphi \tau_+(Y_b) = \tau_+^{-1} \tau_-(X_b^{-1}) = \tau_+^{-1} \tau_- \tau_+^{-1}(X_b^{-1}) = \sigma^{-1}(X_b^{-1}).$$

This gives the first relation from (3.18); the second follow directly from (3.12) and the definition of the series $\Xi(X, \Lambda; q, t)$. The remaining formulas result from (3.14) and the structure of this series. \square

We note that one can prove this proposition *directly* using the relation

$$\sigma^{-1}(X_b) = \sigma^{-2}(\sigma(X_b)) = \sigma^{-2}(Y_b^{-1}) = T_{w_0} Y_{w_0(b)} T_{w_0}^{-1}$$

from Proposition 3.2.2 from [C8]. Combined with (2.43), which states that

$$E_b^* = C_b T_{w_0}(E_{\varsigma(b)}) \text{ for } C_b = \prod_{\nu \in \nu_R} t_\nu^{l_\nu(u_b) - \frac{l_\nu(w_0)}{2}} = C_{\varsigma(b)}, \varsigma(b) = -w_0(b),$$

we obtain that

$$(3.21) \quad \sigma^{-1}(X_a)(E_b^*) = q^{(a_\sharp, b)}(E_b^*) \quad \text{for } a, b \in B.$$

Indeed,

$$\begin{aligned} C_b^{-1} \sigma^{-1}(X_a)(E_b^*) &= T_{w_0} Y_{w_0(a)} T_{w_0}^{-1} (T_{w_0}(E_{\varsigma(b)})) = T_{w_0} (Y_{w_0(a)}(E_{\varsigma(b)})) \\ &= C_{\varsigma(b)}^{-1} q^{-(w_0(a), \varsigma(b)_\sharp)}(E_b^*) = C_b^{-1} q^{(a, b_\sharp)}(E_b^*). \end{aligned}$$

This provides a direct approach to the justification the symmetries (3.12), though related to that based on the interpretation of $G(X, \Lambda)$ as the reproducing kernel of the DAHA-Fourier transform.

The symmetry (3.12) coupled with (3.20) becomes even simpler for the standard Ψ -intertwiners (without the conjugation by τ_+^{-1}). Following (2.56), let $\mathfrak{G}'_i = \Psi'_i \psi_i^{-1}$ ($i \geq 0$) for $\psi_i = t_i^{1/2} + (t_i^{1/2} - t_i^{-1/2})/(Y_{\alpha_i} - 1)$. We also set $\mathfrak{G}'_{\pi_r} = \Psi'_{\pi_r}$ ($r \in O$) and define $\mathfrak{G}'_{\widehat{w}}$ accordingly.

Then for $\widehat{w} = aw \in \widehat{W}^b$, where $a \in B$, $w \in W$,

$$(3.22) \quad \begin{aligned} (\mathfrak{G}'_{\widehat{w}^{-1}})^\lambda (G(X, \Lambda)) &= \widehat{w}^x (G(X, \Lambda)) \\ &= q^{\frac{a^2}{2}} X_{-a} G(E_b^*(X) \mapsto \widehat{w}(E_b^*(X)), \Lambda). \end{aligned}$$

Indeed, $\varphi(\tau_-^{-1} \tau_+(H)) = \varphi(\tau_-^{-1} \varphi \tau_+(H)) = H$ for $H = T_i$ ($i \geq 0$) or for $H = \pi_r$ ($r \in O$). Therefore

$$\varphi(\Psi'_i) = T_i + \frac{t_i^{1/2} - t_i^{-1/2}}{X_{\alpha_i} - 1} = \frac{t_i^{1/2} X_{\alpha_i} - t^{-1/2}}{X_{\alpha_i} - 1} s_i, \quad \varphi(\mathfrak{G}'_i) = s_i$$

$$\text{for } 0 \leq i \leq n \quad \text{and} \quad \varphi(\Psi'_{\pi_r}) = \varphi(\mathfrak{G}'_{\pi_r}) = \pi_{r^*} \quad (r \in O).$$

Formula (3.22) is compatible with the Shintani-type formula (3.11). As a matter of fact, this provides a direct way for establishing (3.11). Thus the formulas for the action of the Ψ -intertwiners on the E -polynomials are essentially sufficient for a direct verification of Theorem 3.1.

3.3. W -spinors. By W -spinors, we simply mean maps $W \rightarrow \mathcal{V}$; we denote the space of W -spinors by $\text{Spin}(\mathcal{V})$. Thus $\text{Spin}(\mathcal{V})$ is naturally a $\mathbb{Q}'_{q,t}$ -algebra under pointwise multiplication and addition. For any $w \in W$, denote by ζ_w the characteristic function $\zeta_w(u) = \delta_{wu}$. These are pairwise orthogonal idempotents in $\text{Spin}(\mathcal{V})$. Any element in $\text{Spin}(\mathcal{V})$ can be written uniquely as

$$(3.23) \quad f = \sum_{w \in W} f_w \zeta_w, \quad \text{where } f_w \stackrel{\text{def}}{=} f(w) \in \mathcal{V}.$$

We refer to f_w as the w -component of f .

We equip $\text{Spin}(\mathcal{V})$ with an action of W via

$$(3.24) \quad (\delta(w)f)(u) \stackrel{\text{def}}{=} f(w^{-1}u).$$

Note that $\delta(w)(\zeta_v) = \zeta_{wv}$ and hence for any $f \in \text{Spin}(\mathcal{V})$,

$$(3.25) \quad (\delta(v)f)_w = f_{v^{-1}w}.$$

One has a natural embedding

$$(3.26) \quad \delta(f) \stackrel{\text{def}}{=} \sum_{w \in W} f \zeta_w, \quad f \in \mathcal{V};$$

its image is the space of W -invariants of $\text{Spin}(\mathcal{V})$, which will be denoted by $\text{Spin}^\delta(\mathcal{V})$. These W -invariants will be called δ -spinors.

These definitions are completely independent of the natural action of W in \mathcal{V} and can be applied to any spaces of functions instead of \mathcal{V} .

Using the action of W in \mathcal{V} , there is another embedding $\varrho : \mathcal{V} \rightarrow \text{Spin}(\mathcal{V})$ defined by

$$(3.27) \quad \varrho(f) \stackrel{\text{def}}{=} \sum_{w \in W} w^{-1}(f) \zeta_w.$$

The spinors in its image will be called *principal spinors* or ϱ -spinors, sometimes, simply *functions*.

For W -invariant $f \in \mathcal{V}$, the spinors $\varrho(f)$ and $\delta(f)$ coincide; thus we will simply write f for W -invariant functions. For arbitrary $f \in \mathcal{V}$ we may also write $f^\varrho \stackrel{\text{def}}{=} \varrho(f)$.

Spinor difference operators. Generally, any endomorphism of \mathcal{V} acts pointwise in $\text{Spin}(\mathcal{V})$. For instance, for a translation Γ_b , we set

$$(3.28) \quad \Gamma_b(f)(u) \stackrel{\text{def}}{=} \Gamma_b(f(u)).$$

We define

$$(3.29) \quad \delta(\Gamma_b) = \Gamma_b^\delta \stackrel{\text{def}}{=} \sum_{w \in W} \Gamma_b \zeta_w,$$

$$(3.30) \quad \varrho(\Gamma_b) = \Gamma_b^\varrho \stackrel{\text{def}}{=} \sum_{w \in W} \Gamma_{w^{-1}(b)} \zeta_w,$$

where ζ_w acts by multiplication in $\text{Spin}(\mathcal{V})$.

Similarly, we may view X_b as a pointwise multiplication operator:

$$(3.31) \quad X_b(f)(u) \stackrel{\text{def}}{=} X_b(f(u)).$$

By a *spinor difference operator* we mean any linear combination of operators of the form $g(X) \Gamma_b \zeta_w$ ($g(X) \in \mathbb{Q}'_{q,t}(X), b \in B, w \in W$). *Spinor difference-reflection operators* are defined as linear combinations of operators of the form $g(X) \Gamma_b \zeta_w \delta(v)$, where $v \in W$.

Recall that \mathcal{A} denotes the algebra of difference-reflection operators (see the beginning of Section 2). Replacing polynomials by rational functions in the definition of $\text{Spin}(\mathcal{V})$, we define an action of \mathcal{A} in $\text{Spin}(\mathbb{Q}'_{q,t}(X))$ by sending

$$\phi : g \Gamma_b w \mapsto \varrho(g) \varrho(\Gamma_b) \delta(w), \quad \text{where } g \in \mathbb{Q}'_{q,t}(X), b \in B, w \in W.$$

It is a homomorphism of algebras: $\phi : \mathcal{A} \rightarrow \text{End}_{\mathbb{Q}'_{q,t}}(\text{Spin}(\mathbb{Q}'_{q,t}(X)))$.

We obtain an action of \mathcal{H} in $\text{Spin}(\mathbb{Q}'_{q,t}(X))$ by composing ϕ with the polynomial representation viewed as a homomorphism $\mathcal{H} \rightarrow \mathcal{A}$; here \mathcal{H} acts by spinor difference-reflection operators. Note that \mathcal{H} does not preserve $\text{Spin}(\mathcal{V})$.

3.4. Spinor RE-procedure. Following [CM], we set

$$(3.32) \quad \begin{aligned} \mathfrak{ae}^\delta(\mathcal{L}) &\stackrel{\text{def}}{=} \delta(X_{\rho_k} \Gamma_{-\rho_k}) \phi(\mathcal{L}) \delta(X_{\rho_k} \Gamma_{-\rho_k})^{-1}, \\ \mathfrak{ae}^\delta(F) &\stackrel{\text{def}}{=} \delta(X_{\rho_k} \Gamma_{-\rho_k}) (\varrho(F)), \end{aligned}$$

for any function F and any difference-reflection operator \mathcal{L} . The non-symmetric variant of (3.1), δ -Ruijsenaars-Etingof procedure, is then defined as

$$(3.33) \quad RE^\delta(\mathcal{L}) \stackrel{\text{def}}{=} \lim_{k \rightarrow \infty} \mathfrak{ae}^\delta(\mathcal{L}), \quad RE^\delta(F) \stackrel{\text{def}}{=} \lim_{k \rightarrow \infty} \mathfrak{ae}^\delta(F),$$

where $k \rightarrow \infty$ means $\Re k_\nu \rightarrow \infty$ for all ν (equivalently, $t_\nu \rightarrow 0$).

We record here the following formulas for later use:

$$(3.34) \quad \mathfrak{ae}^\delta(v \mathcal{L} w) = v \mathfrak{ae}^\delta(\mathcal{L}) w \quad \text{for all } v, w \in W,$$

$$(3.35) \quad \mathfrak{ae}^\delta(X_b) = \sum_{w \in W} \prod_{\nu} t_{\nu}^{-(\rho_{\nu}^{\vee}, w^{-1}(b))} X_{w^{-1}(b)} \zeta_w,$$

$$(3.36) \quad \mathfrak{ae}^\delta(\Gamma_b) = \sum_{w \in W} \prod_{\nu} t_{\nu}^{-(\rho_{\nu}^{\vee}, w^{-1}(b))} \Gamma_{w^{-1}(b)} \zeta_w.$$

Spinor Whittaker function. Let $\Omega(X, \Lambda) \stackrel{\text{def}}{=} RE_x^\delta(G(X, \Lambda)) / \tilde{\gamma}^\ominus(1)$, where the RE_x is applied to X and we assume that $\rho_k \in B$; for instance,

$\mathbb{Z} \ni k_\nu \rightarrow \infty$ is sufficient for $B = P$. Equivalently, we can introduce

$$G'(X, \Lambda) = G(X, \Lambda) \frac{\tilde{\gamma}^\ominus(X) q^{\frac{x^2}{2}}}{\tilde{\gamma}^\ominus(q^{\rho_k}) q^{\frac{\rho_k^2}{2}}}, \quad \Omega'(X, \Lambda) = RE_x^\delta(G'(X, \Lambda)),$$

where $\Re k_\nu \rightarrow \infty$ for *arbitrary complex* k . Then $\tilde{\gamma}^\ominus(X) q^{\frac{x^2}{2}} \Omega(X, \Lambda) = \Omega'(X, \Lambda)$. Using G' instead of G will not influence the corresponding operators (studied below) acting on this function since $\tilde{\gamma}^\ominus(X) q^{\frac{x^2}{2}}$ is \widehat{W} -invariant. Recall that $|q| < 1$, so $t_\nu \rightarrow 0$ if $\Re k_\nu \rightarrow \infty$. Equivalently,

$$\tilde{\gamma}^\ominus(X) \Omega(X, \Lambda) = \lim_{\Re k_\nu \rightarrow \infty} \Gamma_{-\rho_k}(\tilde{\gamma}^\ominus(X) G(X, \Lambda) / \tilde{\gamma}^\ominus(q^{\rho_k})).$$

Proposition 3.3. *The limit, as $\Re k_\nu \rightarrow \infty$ for all ν , of the series $\Gamma_{-\rho_k}^\delta(\Xi(X, \Lambda; q, t))$ exists; here $\Xi(X, \Lambda; q, t)$ is the series from (3.7). Accordingly, one has*

$$(3.37) \quad \Omega(X, \Lambda) = (\tilde{\gamma}_x^\ominus \tilde{\gamma}_\lambda^\ominus)^{-1} \sum_{b \in B} q^{(b, b)/2} \frac{\overline{E}_b(\Lambda)}{\langle \overline{E}_b, \overline{E}_b \rangle_\circ} \sum_{w \in W} a_{b, w} X_{-b_-} \zeta_w,$$

where $a_{b, w}$ is the limit, as all $t_\nu \rightarrow 0$, of the coefficient of $X_{-w(b_-)}$ in E_b^* . In particular, $a_{b, w} \in \mathbb{Z}[q]$ and $a_{b, u_b^{-1}} = 1$, $a_{b, \text{id}} = \delta_{b, b_-}$.

Proof. First, one has

$$\delta(X_{\rho_k} \Gamma_{-\rho_k}) (\tilde{\gamma}_x^\ominus)^{-1} = q^{(\rho_k, \rho_k)/2} (\tilde{\gamma}_x^\ominus)^{-1} \delta(\Gamma_{-\rho_k}),$$

as operators acting on spinor-functions of X (due to the W -invariance of $\tilde{\gamma}^\ominus$, we omit ϱ here). Hence it suffices to consider the limit of

$$q^{-(b_-, \rho_k)} \delta(\Gamma_{-\rho_k}) \cdot \varrho(E_b^*(X)),$$

or equivalently, for each $w \in W$, the limit of

$$(3.38) \quad q^{-(b_-, \rho_k)} \Gamma_{-\rho_k}(w^{-1}(E_b^*)) \zeta_w.$$

Using (2.13), the limit of (3.38) as $t_\nu \rightarrow 0$ clearly exists and is given as $a_{b, w} X_{-b_-} \zeta_w$ for $a_{b, w}$ as claimed. By (2.41), $\overline{E}_b^* = (\overline{E}_b^\dagger)^*$. Hence Corollary 2.6 implies that $a_{b, w} \in \mathbb{Z}[q]$. \square

This proposition can be used to justify the existence of the RE -limits of the Dunkl operators, which we will call *Toda-Dunkl operators*. Moreover, We arrive at the following Whittaker counterpart of Parts (ii) and (iii) of Theorem 3.1 and formula (3.12).

3.5. Main theorem.

Theorem 3.4. (i) The operators $RE^\delta(H^\varphi)$ are well defined for $H \in \dot{\mathcal{H}}^b$, where $H^\varphi = \varphi(H)$; their coefficients are from $\mathbb{Q}'_q[X_b, b \in B]$, so they preserve $\text{Spin}(\overline{V}) \stackrel{\text{def}}{=} \text{Spin}(\mathbb{Q}'_q[X_b, b \in B])$. For instance, the following operators are well defined:

$$(3.39) \quad \widehat{Y}_b \stackrel{\text{def}}{=} RE^\delta(Y_b), \quad \widehat{X}_b \stackrel{\text{def}}{=} RE^\delta(\widetilde{X}_b) \text{ for } \widetilde{X}_b \stackrel{\text{def}}{=} \ddot{Y}_{-b}^\varphi = t^{(b_+, \rho^\vee)} X_b, \\ \widehat{T}_i \stackrel{\text{def}}{=} RE^\delta(\ddot{T}_i) \text{ for } i > 0, \quad \widehat{T}_0 \stackrel{\text{def}}{=} RE^\delta(\ddot{T}_0^\varphi), \quad \widehat{\pi}_r \stackrel{\text{def}}{=} RE^\delta(\pi_r^\varphi) \text{ for } r \in O'.$$

(ii) The function $\Omega(X, \Lambda)$ has the following symmetries:

$$(3.40) \quad \widehat{T}_i(\Omega(X, \Lambda)) = T_i^\lambda(\Omega(X, \Lambda)), \quad \widehat{\pi}_r(\Omega(X, \Lambda)) = \pi_r^\lambda(\Omega(X, \Lambda)) \\ \text{for } 0 \leq i \leq n, r \in O', \quad T_i^\lambda = T_i|_{X \mapsto \Lambda}, \quad \pi_r^\lambda = \pi_r|_{X \mapsto \Lambda}.$$

It satisfies the following limiting version of the relations from (3.10):

$$(3.41) \quad \widehat{Y}_b(\Omega(X, \Lambda)) = \Lambda_b^{-1} \Omega(X, \Lambda), \quad \overline{Y}_b^\lambda(\Omega(X, \Lambda)) = \widehat{X}_{-b} \Omega(X, \Lambda),$$

where $\overline{Y}_b^\lambda = \overline{Y}_b|_{X \mapsto \Lambda}$ for $b \in B$.

(iii) For an arbitrary $c \in B$, let $f(q^{c^\dagger}) \stackrel{\text{def}}{=} f_{u^{-1}}(q^{c^-})$ for a spinor $f = \sum_{w \in W} f_w \zeta_w$ and for $u = u_c \in W$ from Proposition 1.2 such that $u(c) = c_- \in B_-$ and $l(u)$ is minimal possible. Then

$$(3.42) \quad \widetilde{\gamma}^\ominus(1) \Omega(q^{c^\dagger}, \Lambda) = \overline{E}_c(\Lambda) \prod_{i=1}^n \prod_{j=1}^\infty \frac{1}{1 - q_i^j}, \quad \widetilde{\gamma}^\ominus(1) = \sum_{b \in B} q^{b^2/2}.$$

Equivalently, see (3.37) and (2.42),

$$(3.43) \quad \sum_{b \in B} q^{(b_- - c_-)^2/2} \frac{\overline{E}_b(\Lambda)}{\langle \overline{E}_b, \overline{E}_b \rangle_\circ} a_{b, u_c^{-1}} = \widetilde{\gamma}_\lambda^\ominus \overline{E}_c(\Lambda) \prod_{i=1}^n \prod_{j=1}^\infty \frac{1}{1 - q_i^j}.$$

Proof. The key claim here is (ii). It follows from Theorem 3.1 and provides the existence of the operators in (i). Let us demonstrate this in the case of \widehat{Y}_b . One uses (3.10) as follows:

$$(3.44) \quad Y_b(q^{\frac{(c_\sharp^\dagger, c_\sharp^\dagger)}{2} - (\rho_k, \rho_k)}) E_c^\star(X) (\widetilde{\gamma}_x^\ominus)^{-1} = \langle Y_b(G(X, \Lambda)) E_c^\star(\Lambda) \widetilde{\gamma}_\lambda^\ominus \mu_\circ(\Lambda) \rangle, \\ = \langle \Lambda_b^{-1} G(X, \Lambda) E_c^\star(\Lambda) \widetilde{\gamma}_\lambda^\ominus \mu_\circ(\Lambda) \rangle.$$

Applying \mathfrak{ae}^δ and taking $t_\nu \rightarrow 0$, the right-hand side of (3.44) is well defined. Hence it follows that the action of $RE^\delta(Y_b)$ is well defined on

the spinor

$$RE^\delta(q^{-(c_-, \rho_k) - \frac{(\rho_k, \rho_k)}{2}} E_c^\star(X) (\tilde{\gamma}_x^\ominus)^{-1}) = \left(\sum_w a_{c,w} X_{-c_-} \zeta_w \right) (\tilde{\gamma}_x^\ominus)^{-1}.$$

In general, one obtains that the action of the operators from (i) is well defined when they are applied to linear combinations of spinors of the form $X_b \zeta_w (\tilde{\gamma}_x^\ominus)^{-1}$ for dominant regular b and $w \in W$. The operators $\mathfrak{ae}^\delta(H)$ have rational coefficients; nevertheless, this property is sufficient to see that their coefficients are well defined in the limit $t_\nu \rightarrow 0$. Moreover, this gives that the coefficients of $RE^\delta(H)$ actually belong to $\mathbb{Q}'_q[X_b, b \in B]$, i.e., do not have nontrivial denominators. As a matter of fact, this can be formally deduced from (2.7) (see also (4.4)) and the fact that $RE^\delta(H)$ exist.

We will give below a direct and constructive proof of the existence of these operators and the absence of the denominators. Actually, we will prove a stronger result based on direct calculation of the RE^δ -limits of the Dunkl operators, which, for instance, allows to obtain the formulas for the leading terms of the spinor Toda-Dunkl operators and clarify their structure (including the analysis of the vanishing coefficients).

The claims in (iii) follow directly from Theorem 3.1. This is a non-symmetric generalization of the q -Shintani formulas, which in turn generalize the classical Shintani-type formulas in the theory of p -adic Whittaker functions. Note that the identity from (3.43) does not contain/require spinors; it involves only the \overline{E} -polynomials and the coefficients $\{a_{b,u}\}$. \square

Symmetrization. The symmetric (nonspinor) q -Whittaker function $\mathcal{W}(X, \Lambda)$ constructed in Theorem 3.2 of [C10] is the symmetrization of $\Omega(X, \Lambda)$. More precisely, one has

$$(3.45) \quad \delta(\mathcal{W}(X, \Lambda)) = \sum_{w \in W} \hat{T}_w(\Omega(X, \Lambda)) = \sum_{w \in W} T_w^\lambda(\Omega(X, \Lambda)).$$

In particular, the right-hand side is a diagonal spinor (in the image of δ); all its components coincide. See Propositions 5.6 and 5.5 below.

We note that the (nonaffine) hat-symmetrizer $\sum_{w \in W} \hat{T}_w$ preserves the id -component of any W -spinor, which can be readily deduced from (3.35) and formulas for \ddot{T}_i acting in the polynomial representation. See (5.26) below for explicit formulas for $\hat{T}_i (i > 0)$, which are sufficient to check this claim. Therefore it is not actually necessary to perform the symmetrization in (3.45) because the id -component of $\Omega(X, \Lambda)$ is

exactly $\mathcal{W}(X, \Lambda)$. Using (3.42) and formulas (3.40) for \widehat{T}_i with $i > 0$, we see that this coincidence can be also deduced from formula (2.34), which states that $\overline{E}_b = \overline{P}_b$ for $b \in B_-$.

Explicitly, one has

$$(3.46) \quad \mathcal{W}(X, \Lambda) = (\tilde{\gamma}_x^\ominus \tilde{\gamma}_\lambda^\ominus)^{-1} \sum_{b \in B_-} q^{(b,b)/2} \frac{X_{b+} \overline{P}_b(\Lambda^{-1})}{\prod_{i=1}^n \prod_{j=1}^\infty (\alpha_i^\vee, b) (1 - q_i^j)},$$

$$\tilde{\gamma}^\ominus(1) \mathcal{W}(q^c, \Lambda) = \overline{P}_c(\Lambda) \prod_{i=1}^n \prod_{j=1}^\infty \frac{1}{1 - q_i^j} \quad \text{for } c \in B_-,$$

$$(3.47) \quad \sum_{b \in B_-} \frac{q^{(b-c)^2/2} \overline{P}_b(\Lambda)}{\prod_{i=1}^n \prod_{j=1}^\infty (\alpha_i^\vee, b) (1 - q_i^j)} = \tilde{\gamma}_\lambda^\ominus \overline{P}_c(\Lambda) \prod_{i=1}^n \prod_{j=1}^\infty \frac{1}{1 - q_i^j}.$$

The latter formula results from (3.43) and the equality $\overline{E}_b = \overline{P}_b$ for $b \in B_-$. Indeed, when $c \in B_-$ and hence $u_c = \text{id}$, the coefficient $a_{b, u_c^{-1}}$ is nonzero only for $b \in B_-$; in this case, one has $a_{b, \text{id}} = 1$ and the summation in (3.47) ranges over $b \in B_-$.

It is worth mentioning that the left-hand side of (3.47) becomes zero in the (p -adic) limit $q \rightarrow 0$ when $c \notin B_-$, which gives that

$$(3.48) \quad \lim_{q \rightarrow 0} \mathcal{W}(q^c, \Lambda) = 0 \quad \text{unless } c \in B_-.$$

3.6. Fourier transform. Let us interpret the function $\Omega(X, \Lambda)$ as the reproducing kernel of the nil-DAHA Fourier transform.

The $X \leftrightarrow Y$ symmetry of \mathcal{H}^b is not present in $\overline{\mathcal{H}}^b$. More precisely, the anti-involution φ from (1.43) does not act in $\overline{\mathcal{H}}^b$. To recover this symmetry, we define $\check{\mathcal{H}}^{b, \varphi} \stackrel{\text{def}}{=} \varphi(\check{\mathcal{H}}^b)$, a $\check{\mathbb{Q}}'_{q,t}$ -subalgebra of \mathcal{H}^b ; see (1.34) for the definition of $\check{\mathbb{Q}}'_{q,t}$.

As an abstract algebra, $\check{\mathcal{H}}^{b, \varphi}$ can be described as follows. Let $\check{\Pi}^b \stackrel{\text{def}}{=} \varphi(\Pi^b)$, which is isomorphic to the abelian group Π^b , and let $\check{\pi}_r = \varphi(\pi_r)$ for $r \in O'$, $\check{T}_i = \varphi(\check{T}_i)$ for $0 \leq i \leq n$. Note that $\check{T}_i = \check{T}_i$ unless $i = 0$. We set $\check{T}'_i \stackrel{\text{def}}{=} \varphi(\check{T}'_i)$, i.e., $\check{T}'_i = \check{T}_i - (t_i - 1)$. Then in terms of the generators

$$\check{\Pi}^b, \quad \check{T}_i \quad (0 \leq i \leq n), \quad Y_b \quad (b \in B),$$

the defining relations for $\check{\mathcal{H}}^{b, \varphi}$ are:

- (o) $(\check{T}_i - t_i)(\check{T}_i + 1) = 0$, for $0 \leq i \leq n$;
- (i) $\check{T}_i \check{T}_j \check{T}_i \cdots = \check{T}_j \check{T}_i \check{T}_j \cdots$, m_{ij} factors on each side;

- (ii) $\check{\pi}_r^{-1} \check{T}_i \check{\pi}_r = \check{T}_j$ if $\pi_r(\alpha_i) = \alpha_j$, $\pi_r \in \Pi^b$;
- (iii) $\check{T}_i Y_b = Y_b Y_{\alpha_i}^{-1} \check{T}_i$ if $(b, \alpha_i^\vee) = 1$, $0 \leq i \leq n$;
- (iv) $\check{T}_i Y_b = Y_b \check{T}_i$ if $(b, \alpha_i^\vee) = 0$, $0 \leq i \leq n$;
- (v) $\check{\pi}_r^{-1} Y_b \check{\pi}_r = Y_{\pi_r(b)} = Y_{u_r^{-1}(b)} q^{(\omega_r^*, b)}$.

The elements \tilde{X}_b ($b \in B$) from (3.39) belong to $\dot{\mathcal{H}}^{b, \varphi}$; they are the images of \check{Y}_{-b} under φ . Namely, $X_b = \varphi(Y_{-b})$ and $\tilde{X}_b \stackrel{\text{def}}{=} \varphi(\check{Y}_{-b}) = q^{(b_+, \rho_k)} X_b$; in the last equality, we use that $(-b)_+ = -w_0(b_+)$ and $(-w_0(b_+), \rho) = (b_+, \rho)$. The corresponding relations in $\dot{\mathcal{H}}^{b, \varphi}$ are obtained by applying φ to those from $\dot{\mathcal{H}}^b$.

By construction, the anti-involution φ of \mathcal{H}^b sends $\dot{\mathcal{H}}^b$ to $\dot{\mathcal{H}}^{b, \varphi}$. The automorphism σ from (1.50) also has this property. Explicitly, σ preserves q and

$$(3.49) \quad \begin{aligned} \sigma : \check{T}_i &\mapsto \check{T}_i \ (i \geq 0), \ X_b \mapsto Y_b^{-1}, \\ \pi_r &\mapsto X_{\omega_r} T_{u_r^{-1}} = \varphi(\pi_r^{-1}) = \check{\pi}_r^{-1}. \end{aligned}$$

Now define $\overline{\mathcal{H}}^{b, \varphi}$ to be the specialization of $\dot{\mathcal{H}}^{b, \varphi}$ for all $t_\nu = 0$. The anti-isomorphism φ and the automorphism σ are compatible with this specialization, and we use the same symbols to denote the resulting maps from $\overline{\mathcal{H}}^b$ to $\overline{\mathcal{H}}^{b, \varphi}$.

Recall the definition of the algebra $\overline{\mathcal{H}}^{b, \dagger}$ from Section 2.5. The involution ε of \mathcal{H}^b from (1.51) conjugates q, t_ν . Accordingly, we obtain an isomorphism $\varepsilon^\dagger : \overline{\mathcal{H}}^{b, \dagger} \rightarrow \overline{\mathcal{H}}^{b, \varphi}$ given by

$$(3.50) \quad \begin{aligned} \varepsilon^\dagger : \overline{T}_i^\dagger &\mapsto \overline{T}_i' \ (i > 0), \ X_b \mapsto Y_b, \ q \mapsto q^{-1}, \\ \pi_r &\mapsto \check{\pi}_r^{-1} = X_{\omega_r} T_{u_r^{-1}} = \varphi(\pi_r^{-1}). \end{aligned}$$

Now we are ready to state the Fourier transform interpretation of Theorem 3.4.

Corollary 3.5. *Define the transforms*

$$(3.51) \quad \mathbf{S}(f)(X) \stackrel{\text{def}}{=} \langle f(\Lambda) \Omega(X, \Lambda) \overline{\mu}_\circ \rangle_\circ,$$

$$(3.52) \quad \mathbf{E}(f)(X) \stackrel{\text{def}}{=} \langle f(\Lambda)^* \Omega(X, \Lambda) \overline{\mu}_\circ \rangle_\circ,$$

acting from functions of Λ to X -spinors. Then one has:

$$(3.53) \quad \mathbf{S}((\overline{E}_b^\dagger)^* \tilde{\gamma}_\lambda^\ominus) = \mathbf{E}(\overline{E}_b^\dagger \tilde{\gamma}_\lambda^\oplus) = q^{(b, b)/2} (\tilde{\gamma}_x^\ominus)^{-1} \sum_{w \in W} a_{b, w} X_{-b_-} \zeta_w,$$

and

$$(3.54) \quad \mathbf{S}(H(f)) = \sigma(H)\mathbf{S}(f) \text{ for } H \in \overline{\mathcal{H}}^b, \sigma(H) \in \overline{\mathcal{H}}^{b,\varphi},$$

$$(3.55) \quad \mathbf{E}(H(f)) = \varepsilon^\dagger(H)\mathbf{E}(f) \text{ for } H \in \overline{\mathcal{H}}^{b,\dagger}, \varepsilon^\dagger(H) \in \overline{\mathcal{H}}^{b,\varphi},$$

provided the existence of the transforms $\mathbf{S}(f)$ and $\mathbf{E}(f)$.

Proof. The formula (3.53) is immediate from the explicit expression (3.37) for Ω and the orthogonality relations (2.42). The intertwining properties (3.54) and (3.55) follow from the relations $\varepsilon = \varphi \star$ and $\sigma = \varepsilon \eta = \varphi \star \eta$. \square

Pseudo-polynomial representation. The transform \mathbf{S} embeds $\overline{\mathcal{V}}_\lambda \tilde{\gamma}_\lambda^\ominus$, which is $\overline{\mathcal{V}} \tilde{\gamma}^\ominus$ under $X \mapsto \Lambda$, into the space of X -spinors. The image $\mathbf{S}(\overline{\mathcal{V}}_\lambda \tilde{\gamma}_\lambda^\ominus)$ can be described explicitly as follows.

We need the following properties of the coefficients $a_{b,w}$ from (3.37):

$$(3.56) \quad a_{b,w} = 0 \quad \text{unless} \quad w \geq u_b^{-1},$$

$$(3.57) \quad a_{b,w} = a_{b,wy} \quad \text{if} \quad y \in W^{b-}.$$

Recall that W^b is the centralizer of b in W . These properties are immediate from the description of the $a_{b,w}$ given in Proposition 3.3. For (3.56), one must also use (2.14). Formula (3.57) asserts that $a_{b,w}$ depends only on the coset of w in W/W^{b-} . The elements u_b^{-1} for $b \in W(b_-)$ are exactly the minimum length coset representatives for W/W^{b-} .

For any $b \in B$, we set

$$(3.58) \quad \mathbf{X}_b = \sum_{y \in u_b^{-1}W_{b-}} X_{-b-} \zeta_y = X_{-b-} \sum_{y \in u_b^{-1}W_{b-}} \zeta_y.$$

Then we can write (3.53) as follows:

$$(3.59) \quad \mathbf{S}((\overline{E}_b^\dagger)^* \tilde{\gamma}_\lambda^\ominus) = q^{(b,b)/2} (\tilde{\gamma}_x^\ominus)^{-1} \sum_{\substack{c \in W(b) \\ u_c^{-1} \geq u_b^{-1}}} a_{b,u_c^{-1}} \mathbf{X}_c,$$

We conclude that the image $\mathbf{S}(\overline{\mathcal{V}}_\lambda \tilde{\gamma}_\lambda^\ominus)$ is precisely the span of the elements $\{\mathbf{X}_b \tilde{\gamma}_x^\ominus, b \in B\}$.

Note that when $b \in B$ is regular, i.e., $W^{b-} = \{\text{id}\}$, then one simply has $\mathbf{X}_b = X_{-b-} \zeta_{u_b^{-1}}$. Hence the image of \mathbf{S} contains all elements $X_b \zeta_w \tilde{\gamma}_x^\ominus$ for dominant regular b and $w \in W$. This observation was used above in the proof of Theorem 3.4 to establish the existence of

the limits $RE^\delta(H)$ for all $H \in \dot{\mathcal{H}}^{b,\varphi}$ as spinor difference-reflection operators; in particular, it establishes the existence of \widehat{Y}_b for all $b \in B$, which we call the *Toda-Dunkl operators*.

The remainder of the paper is devoted to a direct proof of the existence of these operators, which is quite interesting in its own right and provides valuable information about the structure of these operators.

4. MANAGING G-PRODUCTS

We will give in the next two sections a constructive justification of the existence of the Toda-Dunkl operators and the other operators $RE^\delta(H)$ for $H \in \dot{\mathcal{H}}^{b,\varphi}$, without any reference to the spinor Whittaker function $\Omega(X, \Lambda)$ obtained above. We assume that $B = P$ for the remainder of the paper, which is sufficient for establishing the existence and finding the formulas.

The direct approach involves some nontrivial combinatorics of reduced decompositions and λ -sequences in \widehat{W} but gives more exact information about the structure and the coefficients of such operators. We will begin with basic estimates and examples and then we will proceed by induction in the next section.

4.1. Basic t -estimates. We will denote $\text{Spin}(\mathbb{Q}'_{q,t}(X))$ by $\text{Spin}(\mathcal{V}')$ in this and further sections. For any $f \in \text{Spin}(\mathcal{V}')$, $w \in W$, and $\nu \in \nu_R$, we define $\text{ord}_w^\nu(f)$ to be the order of the w -component f_w with respect to t_ν . Hence for $f, g \in \text{Spin}(\mathcal{V}')$,

$$(4.1) \quad \text{ord}_w^\nu(fg) = \text{ord}_w^\nu(f) + \text{ord}_w^\nu(g).$$

Note that

$$(4.2) \quad \text{ord}_w^\nu(\delta(v)f) = \text{ord}_{v^{-1}w}^\nu(f),$$

due to (3.25).

For $f \in \text{Spin}(\mathcal{V}')$ and $v, w \in W$, define

$$(4.3) \quad \text{ord}_w^\nu(fv) \stackrel{\text{def}}{=} \text{ord}_w^\nu(f), \quad {}^\dagger\text{ord}_w^\nu(vf) \stackrel{\text{def}}{=} \text{ord}_w^\nu(f).$$

More generally, we set $\text{ord}_w^\nu(Aw) = \text{ord}_w^\nu(A) = {}^\dagger\text{ord}_w^\nu(wA)$ for $w \in W$ and any spinor difference operators A , the sums of the products of X -spinors and Γ -spinors.

Recall that given a reduced decomposition $\widehat{u} = \pi_r s_{j_l} \cdots s_{j_1}$ for $l = l(\widehat{u})$ and $r \in O$,

$$(4.4) \quad \begin{aligned} T_{\widehat{u}} &= \widehat{u} G_{\widetilde{\alpha}^l} \cdots G_{\widetilde{\alpha}^1} = G_{-\widetilde{\beta}^l} \cdots G_{-\widetilde{\beta}^1} \widehat{u} \\ \text{for } \widetilde{\alpha}^1 &= \alpha_{j_1}, \widetilde{\alpha}^2 = s_{j_1}(\alpha_{j_2}), \dots, \widetilde{\beta}^r = -b(\widetilde{\alpha}^r) \in \widetilde{R}_+, \end{aligned}$$

$$(4.5) \quad \begin{aligned} G_{\widetilde{\alpha}} &\stackrel{\text{def}}{=} t_{\alpha}^{1/2} + \frac{t_{\alpha}^{1/2} - t_{\alpha}^{-1/2}}{X_{\widetilde{\alpha}}^{-1} - 1} (1 - s_{\widetilde{\alpha}}) \\ &= \frac{t_{\alpha}^{1/2} X_{\widetilde{\alpha}}^{-1} - t_{\alpha}^{-1/2}}{X_{\widetilde{\alpha}}^{-1} - 1} - \frac{t_{\alpha}^{1/2} - t_{\alpha}^{-1/2}}{X_{\widetilde{\alpha}}^{-1} - 1} s_{\widetilde{\alpha}}, \\ G_{-\widetilde{\alpha}} &= \frac{t_{\alpha}^{1/2} X_{\widetilde{\alpha}} - t_{\alpha}^{-1/2}}{X_{\widetilde{\alpha}} - 1} - \frac{t_{\alpha}^{1/2} - t_{\alpha}^{-1/2}}{X_{\widetilde{\alpha}} - 1} s_{\widetilde{\alpha}}, \end{aligned}$$

where $\widetilde{\alpha} \in \widetilde{R}$. Recall that $X_{\widetilde{\alpha}} = X_{\alpha} q^j$ for $\widetilde{\alpha} = [\alpha, j]$ and that $\{\widetilde{\alpha}^1, \widetilde{\alpha}^2, \dots, \widetilde{\alpha}^l\} = \lambda(\widehat{u}) \subset \widetilde{R}_+$. Note that $\widehat{u} s_{\widetilde{\alpha}^l} \cdots s_{\widetilde{\alpha}^1} = \pi_r$.

We will restrict ourselves to the estimates of the orders of operators Y_b^{-1} for $b \in P_+$. This is totally parallel to the case of Y_b and complementary to the proof of Proposition 5.1 provided below, where the case of positive powers will be addressed.

Accordingly,

$$(4.6) \quad T_{\widehat{u}}^{-1} = \widetilde{G}_{\widetilde{\alpha}^1} \cdots \widetilde{G}_{\widetilde{\alpha}^l} \widehat{u}^{-1}, \quad \widetilde{G}_{\widetilde{\alpha}} = \frac{t_{\alpha} - X_{\widetilde{\alpha}}^{-1}}{1 - X_{\widetilde{\alpha}}^{-1}} + \frac{t_{\alpha} - 1}{1 - X_{\widetilde{\alpha}}^{-1}} s_{\widetilde{\alpha}}.$$

where, as above, $\widetilde{\alpha}^1 = \alpha_{j_1}$, $\widetilde{\alpha}^2 = s_{j_1}(\alpha_{j_2})$, $\widetilde{\alpha}^3 = s_{j_1} s_{j_2}(\alpha_{j_3})$ and so on. We set

$$(4.7) \quad \widetilde{f}_{\widetilde{\alpha}} = \frac{t_{\alpha} - X_{\widetilde{\alpha}}^{-1}}{1 - X_{\widetilde{\alpha}}^{-1}}, \quad \widetilde{g}_{\widetilde{\alpha}} = \frac{t_{\alpha} - 1}{1 - X_{\widetilde{\alpha}}^{-1}}, \quad \widetilde{G}_{\widetilde{\alpha}} = \widetilde{f}_{\widetilde{\alpha}} + \widetilde{g}_{\widetilde{\alpha}} s_{\widetilde{\alpha}}.$$

Our aim is to estimate the orders of operators $\mathfrak{a}^{\delta}(T_{\widehat{u}})$ with respect to t_{ν} for $\nu \in \nu_R$ aiming at the limit $t_{\nu} \rightarrow 0$ for all ν . Recall that for $b \in P$ and $\nu \in \nu_R$,

$$(4.8) \quad \begin{aligned} q^{(\rho_k, b)} &= q^{\sum_{\nu} k_{\nu}(\rho_{\nu}, b)} = \prod_{\nu} q_{\nu}^{k_{\nu}(\rho_{\nu}^{\vee}, b)} = \prod_{\nu} t_{\nu}^{(\rho_{\nu}^{\vee}, b)}, \\ \text{also, } t^{l(\widehat{w})/2} &= \prod_{\nu} t_{\nu}^{l_{\nu}(\widehat{w})/2} = t_{\nu}^{|\lambda_{\nu}(\widehat{w})|/2} \quad \text{for } \widehat{w} \in \widehat{W}. \end{aligned}$$

The following straightforward formulas are actually the key. For any $\nu \in \nu_R$, $\tilde{\alpha} = [\alpha, \nu_\alpha j]$ and $\delta_{\alpha, \nu} \stackrel{\text{def}}{=} \delta_{\nu_\alpha, \nu}$,

$$(4.9) \quad \text{ord}_w^\nu(\mathfrak{ae}^\delta(\tilde{g}_{\tilde{\alpha}})) = \begin{cases} 0, & \text{if } w^{-1}(\alpha) > 0, \\ -(w^{-1}(\alpha), \rho_\nu^\vee), & \text{if } w^{-1}(\alpha) < 0, \end{cases}$$

and

$$(4.10) \quad \text{ord}_w^\nu(\mathfrak{ae}^\delta(\tilde{f}_{\tilde{\alpha}})) = \begin{cases} \delta_{\alpha, \nu}, & \text{if } w^{-1}(\alpha) > 0, \\ 0, & \text{if } w^{-1}(\alpha) < 0. \end{cases}$$

Indeed,

$$(4.11) \quad \mathfrak{ae}^\delta(\tilde{g}_{\tilde{\alpha}}) = \sum_{w \in W} \frac{t_\nu - 1}{1 - q^{(\rho_k, w^{-1}(\alpha))} w^{-1}(X_\alpha^{-1})} \zeta_w,$$

$$(4.12) \quad \mathfrak{ae}^\delta(\tilde{f}_{\tilde{\alpha}}) = \sum_{w \in W} \frac{t_\nu - q^{(\rho_k, w^{-1}(\alpha))} w^{-1}(X_\alpha^{-1})}{1 - q^{(\rho_k, w^{-1}(\alpha))} w^{-1}(X_\alpha^{-1})} \zeta_w.$$

We also need the t_ν -orders of $\Gamma_b = (-b)$:

$$\text{ord}_w^\nu(\mathfrak{ae}^\delta(\Gamma_b)) = \text{ord}_w^\nu\left(\sum_{v \in W} q^{-(\rho_k, v^{-1}(b))} \Gamma_{v^{-1}(b)} \zeta_v\right) = -(\rho_\nu^\vee, w^{-1}(b)).$$

Applying this formula to $s_{\tilde{\alpha}} s_\alpha = \Gamma_\alpha^j$ for $\tilde{\alpha} = [\alpha, \nu_\alpha j] \in \tilde{R}$, we obtain that

$$\text{ord}_w^\nu(\mathfrak{ae}^\delta(s_{\tilde{\alpha}})) = -j(\rho_\nu^\vee, w^{-1}(\alpha)), \quad \text{and}$$

$$(4.13) \quad \text{ord}_w^\nu(\mathfrak{ae}^\delta(\tilde{g}_{\tilde{\alpha}} s_{\tilde{\alpha}})) = \begin{cases} -j(\rho_\nu^\vee, w^{-1}(\alpha)), & \text{if } w^{-1}(\alpha) > 0, \\ -(1+j)(\rho_\nu^\vee, w^{-1}(\alpha)), & \text{if } w^{-1}(\alpha) < 0. \end{cases}$$

Recall that we set $\text{ord}_w^\nu(Aw) = \text{ord}_w^\nu(A) = {}^\dagger\text{ord}_w^\nu(wA)$ for $w \in W$ and any spinor difference operators A , the sums of the products of X -spinors and Γ -spinors. One also has that

$$\tilde{g}_{\tilde{\alpha}} s_{\tilde{\alpha}} = s_\alpha \Gamma^{-j} \tilde{g}_{-\tilde{\alpha}}, \quad \text{and}$$

$$(4.14) \quad {}^\dagger\text{ord}_w^\nu(\mathfrak{ae}^\delta(\tilde{g}_{\tilde{\alpha}} s_{\tilde{\alpha}})) = \begin{cases} (1+j)(\rho_\nu^\vee, w^{-1}(\alpha)), & \text{if } w^{-1}(\alpha) > 0, \\ j(\rho_\nu^\vee, w^{-1}(\alpha)), & \text{if } w^{-1}(\alpha) < 0. \end{cases}$$

4.2. Leading terms. In this section, we provide the estimates for *some* of the coefficients of the Y -operators, which will be main ingredients of the direct justification of the existence (and invertibility) of $RE(Y_b)$ in the next section.

It is straightforward to calculate the top coefficient in the expansion of $\ddot{T}_{\hat{u}}^{-1} = t^{l(\hat{u})/2} T_{\hat{u}}^{-1} \stackrel{\text{def}}{=} \sum_{\hat{w} \in \widehat{W}} \ddot{C}_{\hat{w}} \hat{w}$, which is $\ddot{C}_{\hat{u}^{-1}}$. It can be obtained only by picking the terms without $s_{\tilde{\alpha}}$ from all binomials in (4.6). Thus

$$(4.15) \quad \ddot{C}_{\hat{u}^{-1}} = \prod_{\tilde{\alpha} \in \Lambda(\hat{u})} \tilde{f}_{\tilde{\alpha}} = \prod_{[\alpha, j] \in \Lambda(\hat{u})} \frac{1 - t_{\alpha} q^j X_{\alpha}}{1 - q^j X_{\alpha}}.$$

Let us apply this formula to $\hat{u} = b \in P_+$. Then

$$\lambda_{\nu}(b) = \{\tilde{\alpha} = [\alpha, \nu_{\alpha} j]; \alpha > 0, \nu_{\alpha} = \nu, 0 \leq j < (b, \alpha^{\vee})\}$$

and $(\rho_{\nu}^{\vee} - w(\rho_{\nu}^{\vee}), b)$ is exactly the number of $\tilde{\alpha} \in \lambda_{\nu}(b)$ such that $w^{-1}(\alpha) < 0$, which coincides with the number of $\tilde{f}_{\tilde{\alpha}}$ for $\tilde{\alpha} \in \lambda_{\nu}(b)$ such that $\text{ord}_w^{\nu}(\tilde{f}_{\tilde{\alpha}}) = 0$.

Therefore,

$$(4.16) \quad \begin{aligned} \text{ord}_w^{\nu}(\mathfrak{ae}^{\delta}(\ddot{C}_{-b} \Gamma_b)) &= \sum_{\tilde{\alpha} \in \lambda_{\nu}(b)} \text{ord}_w^{\nu}(\tilde{f}_{\tilde{\alpha}}) - (\rho_{\nu}^{\vee}, w^{-1}(b)) \\ &= 2(\rho_{\nu}^{\vee}, b) - (\rho_{\nu}^{\vee} - w(\rho_{\nu}^{\vee}), b) - (\rho_{\nu}^{\vee}, w^{-1}(b)) = (\rho_{\nu}^{\vee}, b). \end{aligned}$$

We see that the term $q^{-(\rho_k, b)} \mathfrak{ae}^{\delta}(\ddot{C}_{-b} \Gamma_b)$ in the \mathfrak{ae}^{δ} -image of the expansion

$$(4.17) \quad Y_b^{-1} = q^{-(\rho_k, b)} \ddot{Y}_b^{-1} = q^{-(\rho_k, b)} \sum_{\hat{w} \in \widehat{W}} \ddot{C}_{\hat{w}} \hat{w}$$

is of zero order, as it is supposed to be due to the existence and invertibility of $RE(Y_b^{-1})$.

We will prove below that the order $\text{ord}_w^{\nu}(\mathfrak{ae}^{\delta}(\Pi))$ is non-negative for each $w \in W$ and any individual product Π contributing to $Y_b^{\pm 1}$ ($b \in P_+$), where we pick terms with and without s_i from $T_i^{\pm 1}$ in the polynomial representation. Equivalently, we can expand $Y_b^{\pm 1}$ with \hat{u} placed on the left. In the case of Y_b^{-1} , which we mainly consider in this section,

$$\text{ord}_w^{\nu}(\mathfrak{ae}^{\delta}(\Pi'_{\hat{w}})) \text{ord}_w^{\nu}(\mathfrak{ae}^{\delta}(\hat{w})) \geq (\rho_{\nu}^{\vee}, b) \leq \text{ord}_w^{\nu}(\mathfrak{ae}^{\delta}(\Pi''_{\hat{w}})) \text{ord}_w^{\nu}(\mathfrak{ae}^{\delta}(\hat{w}))$$

for any individual products $\Pi'_{\hat{w}}$ of $\tilde{f}_{\tilde{\alpha}j}$ and $\tilde{g}_{\tilde{\alpha}j}$ contributing to $\ddot{C}_{\hat{w}}$ from (4.17) and their counterparts $\Pi''_{\hat{w}}$ for $\dagger \text{ord}$, i.e. for $\ddot{Y}_b^{-1} = \sum \hat{w} \Pi''_{\hat{w}}$ with \hat{w} collected on the left.

4.3. The lowest terms. The smallest possible \widehat{w} that can be obtained from \widehat{u} is when we always pick the terms with $s_{\tilde{\alpha}}$ from the binomials in the product (4.6) for $\ddot{T}_{\widehat{u}}^{-1} = t^{l(\widehat{u})/2} T_{\widehat{u}}^{-1} = \prod_{\nu} t_{\nu}^{l_{\nu}(\widehat{u})/2} T_{\widehat{u}}^{-1}$.

It will contribute to the \ddot{C} -coefficient of π_r^{-1} (maximally distant from \widehat{w}^{-1}). There can be of course other products that contribute to $\ddot{C}_{\pi_r^{-1}}$; their number grows exponentially with $l(\widehat{u})$. This particular product is as follows:

$$(4.18) \quad \prod_{r=1}^l \tilde{g}_{\alpha_{j_r}} = \prod_{r=1}^l \frac{t_{j_r} - 1}{1 - X_{\alpha_{j_r}}^{-1}};$$

recall that $\{\alpha_j\}$ are simple roots.

Let $\nu \in \nu_R$, $w \in W$ and $N_j^{\nu}(b; w)$ be the number of simple reflections s_j for $\nu_j = \nu$ in a given decomposition of $b \in P_+$ such that $w^{-1}(\alpha_j) < 0$. Accordingly, N_0^{ν} is zero when $w^{-1}(\vartheta) < 0$ or $\nu = \nu_{\text{lng}}$ and the number of s_0 in this decomposition of b otherwise. We set $N_j^{\nu}(b)$ if all s_j with $\nu_i = \nu$ are counted, and omit the super-index ν if all $\nu \in \nu_R$ are considered. Actually here we need only the decomposition of $\pi_r^{-1}b$, where $b - \omega_r \in Q$.

In such case, we claim that for any reduced decomposition of $b = b_+$ and for an arbitrary $w \in W$,

$$(4.19) \quad \text{ord}_w^{\nu} \left(q^{-(\rho_k, b)} \mathfrak{ae}^{\delta} \left(\left(\prod_{r=1}^l \tilde{g}_{\alpha_{j_r}} \right) \pi_r^{-1} \right) \right) \geq 0, \quad \text{equivalently,}$$

$$N_0^{\nu}(b; w)(\vartheta, w(\rho_{\nu}^{\vee})) - \sum_{j=1}^n N_j^{\nu}(b; w)(\alpha_j, w(\rho_{\nu}^{\vee})) + (\omega_{r^*}, w(\rho_{\nu}^{\vee})) \geq (b, \rho_{\nu}^{\vee}).$$

Here we use that $\pi_r^{-1} = u_r \Gamma_{b_r}$ and therefore

$$\begin{aligned} \mathfrak{ae}^{\delta}(\pi_r^{-1}) &= u_r \left(q^{-(v^{-1}(\omega_r), \rho_k)} \sum_{v \in W} \Gamma_{v^{-1}(\omega_r)}^{-1} \zeta_v \right) = q^{-(v^{-1}(\omega_r), \rho_k)} \sum_{v \in W} \Gamma_{v^{-1}(\omega_r)}^{-1} \zeta_{u_r v} \\ &= q^{-(w^{-1}u_r(\omega_r), \rho_k)} \sum_{w \in W} \Gamma_{w^{-1}u_r(\omega_r)}^{-1} \zeta_w = q^{(w^{-1}(\omega_{r^*}), \rho_k)} \sum_{w \in W} \Gamma_{w^{-1}(\omega_{r^*})}^{+1} \zeta_w. \end{aligned}$$

Recall that $\omega_{r^*} = -u_r(\omega_r)$ for the image r^* of r under the involution of the nonaffine Dynkin diagram on the set O induced by $-w_0$. Also, $(b, \rho_{\nu}^{\vee}) = (-w_0(b), \rho_{\nu}^{\vee})$ for $\nu \in \nu_R$.

The existence of a reduced decomposition of b such that (4.19) holds for any w can be justified directly using that the left-hand side of (4.19) is additive for $b = a + c$, where $a \in Q \cap P_+$ and $c \in P_+$ (so is its

right-hand side). We use here that $l(a + c) = l(a) + l(c)$ and therefore $N_j^\nu(a + c; w) = N_j^\nu(a; w) + N_j^\nu(c; w)$ if the reduced decomposition of a and c are combined together (without further using the homogeneous Coxeter transformations).

For instance, for $w = \text{id}$ and $b \in P_+$, the inequality becomes the estimate $N_0(b) \geq (b - \omega_r, \rho^\vee) / (\vartheta, \rho^\vee)$. Taking $w = s_j (j > 0)$ and assuming that the root system is not A_1 , we arrive at

$$N_0(b)(\vartheta, \rho^\vee - \alpha_j^\vee) + N_j(b) \geq (b - \omega_r, \rho^\vee) + \delta_{r^*, j}.$$

For $b = \omega_r$, it means that $N_j(\omega_r) \geq \delta_{r^*, j}$ and that $u_r = \pi_r^{-1} \omega_r$ contains at least one s_{r^*} in any of its reduced decompositions. This holds since otherwise $u_r(\omega_{r^*}) = \omega_{r^*}$, which is impossible because $u_r(\omega_{r^*}) = -\omega_r$.

When $w = w_0$, we obtain the inequality, which is of some interest and clarifies the combinatorics related to our estimates:

$$\sum_{j=1}^n N_j(b) = 2(b, \rho^\vee) - N_0(b) \geq (b + \omega_r, \rho^\vee).$$

It can be readily transformed to $N_0(b) \leq (b - \omega_r, \rho^\vee)$ and is sharp for $b = \omega_r$, $r \in O'$.

4.4. Taking one g . Let us now pick only one term $\tilde{g}_{\tilde{\alpha}^i} s_{\tilde{\alpha}^i}$ ($1 \leq i \leq l$) from the binomials in the product (4.6) and then move $\sigma^i \stackrel{\text{def}}{=} s_{\tilde{\alpha}^i}$ to the left. We will also use the notation $s^i \stackrel{\text{def}}{=} s_{\alpha^i}$ for $\tilde{\alpha}^i = [\alpha^i, \nu_{\alpha^i} j]$. Recall that

$$(4.20) \quad T_{\hat{u}}^{-1} = \tilde{G}_{\tilde{\alpha}^1} \cdots \tilde{G}_{\tilde{\alpha}^l} \hat{u}^{-1}, \quad \tilde{G}_{\tilde{\alpha}} = \frac{t_\alpha - X_{\tilde{\alpha}}^{-1}}{1 - X_{\tilde{\alpha}}^{-1}} + \frac{t_\alpha - 1}{1 - X_{\tilde{\alpha}}^{-1}} s_{\tilde{\alpha}}.$$

The contribution of this term to $\ddot{Y}_b^{-1} \stackrel{\text{def}}{=} q^{(\rho_k, b)} Y_b^{-1}$ for $b \in P_+$ is

$$(4.21) \quad \sigma^i \left(\tilde{f}_{\sigma^i(\tilde{\alpha}^1)} \tilde{f}_{\sigma^i(\tilde{\alpha}^2)} \cdots \tilde{f}_{\sigma^i(\tilde{\alpha}^{i-1})} (\tilde{g}_{-\tilde{\alpha}^i}) \tilde{f}_{\tilde{\alpha}^{i+1}} \cdots \tilde{f}_{\tilde{\alpha}^l} \right) \Gamma_b.$$

We claim that the order of this expression is no smaller than (ρ^\vee, b) . Using (4.16) and (4.14), it suffices to check the following lemma.

Lemma 4.1.

$$(4.22) \quad \text{ord}_w^\nu(\tilde{f}_{\sigma^i(\tilde{\alpha}^1)} \tilde{f}_{\sigma^i(\tilde{\alpha}^2)} \cdots \tilde{f}_{\sigma^i(\tilde{\alpha}^{i-1})}) + {}^\dagger \text{ord}_w^\nu(\tilde{g}_{\tilde{\alpha}^i}(\sigma^i)) \\ \geq \text{ord}_w^\nu(\tilde{f}_{\tilde{\alpha}^1} \tilde{f}_{\tilde{\alpha}^2} \cdots \tilde{f}_{\tilde{\alpha}^i}).$$

Let us begin its proof with the example of simplest $w = \text{id}$. We set here and below $\tilde{\alpha}^j = [\alpha^j, \nu_{\alpha^j} m_j]$ for $j = 1, \dots, l$, for instance $\tilde{\alpha}^i = [\alpha^i, \nu_{\alpha^i} m_i]$, and $\tilde{\beta}^r \stackrel{\text{def}}{=} \sigma^i(\tilde{\alpha}^r) = [\beta^r, \nu_{\beta^r} k_r]$ for $r = 1, 2, \dots, i$; thus $\tilde{\beta}^i = \tilde{\alpha}^i$.

If $w = \text{id}$, then $w^{-1}(\alpha^i) = \alpha^i > 0$ and we need to check that for $\tilde{\beta}^r$ such that $\nu_{\beta^r} = \nu$ and for a given $\nu \in \nu_R$,

$$(4.23) \quad (m_i + 1)(\rho_\nu^\vee, \alpha^i) \geq |\{ \tilde{\beta}^r ; \beta^r = s^i(\alpha^r) < 0, 1 \leq r \leq i \}|.$$

Using (1.26) we obtain that if $\alpha^i \neq \beta \in \lambda_\nu(s^i)$, then for each k , either $[\beta, \nu_\beta k]$ or $[\beta', \nu_\beta k]$ for $\beta' = -s^i(\beta)$ can belong to the set $\{\tilde{\beta}^r, 1 \leq r \leq i\}$, but not both.

More exactly, Claim (ii) from Lemma 1.3 gives that there will be exactly one such occurrence for each $0 \leq k \leq (\alpha^i, \beta^\vee) m_i$. The coefficient (α^i, β^\vee) is 1 unless β is short and α^i is long, when it equals ν_{lng} , i.e. it is η_{β, α^i} in the notation from this Lemma.

Therefore the total number \mathfrak{N} of $\tilde{\beta} = [\beta, \nu_\beta k] \in \{\tilde{\beta}^r\}$ such that $\alpha^i \neq \beta \in \lambda(s^i)$ is as follows:

$$\begin{aligned} \mathfrak{N} - (\rho, \alpha^\vee) &= m_i((\rho_{\text{lng}}, \alpha^\vee) - 1 + \nu_{\text{lng}}(\rho_{\text{sht}}, \alpha^\vee)) \\ &= m_i((\rho_{\text{lng}}^\vee, \alpha) - 1 + (\rho_{\text{sht}}, \alpha)) \\ &= m_i((\rho^\vee, \alpha) - 1) \quad \text{for long } \alpha = \alpha^i, \\ \mathfrak{N} - (\rho^\vee, \alpha) &= m_i((\rho_{\text{lng}}^\vee, \alpha) + (\rho_{\text{sht}}, \alpha) - 1) \\ &= m_i((\rho^\vee, \alpha) - 1) \quad \text{for short } \alpha = \alpha^i. \end{aligned}$$

Accordingly, for \mathfrak{N}_ν defined when $\alpha^i \neq \beta \in \lambda_\nu(s^i)$,

$$\begin{aligned} \mathfrak{N}_\nu - (\rho_\nu, \alpha^\vee) &= m_i((\rho_\nu^\vee, \alpha) - \delta_{\alpha, \nu}) \quad \text{for long } \alpha = \alpha^i, \\ \mathfrak{N}_\nu - (\rho_\nu^\vee, \alpha) &= m_i((\rho_\nu^\vee, \alpha) - \delta_{\alpha, \nu}) \quad \text{for short } \alpha = \alpha^i. \end{aligned}$$

Allowing the roots $\{[\alpha, \nu_\alpha k], 0 \leq k \leq m_i\}$, i.e. omitting the restriction $\beta \neq \alpha^i$, we obtain that the right-hand side of (4.23) equals $(1 + m_i)((\rho_\nu^\vee, \alpha^i) - \delta_{\alpha^i, \nu}) + \delta_{\alpha^i, \nu}(1 + m_i)$ for short α^i , which does coincide with the left-hand side, and is strictly smaller than the left-hand side if α^i is long, $(\alpha^i, \rho_{\text{sht}}) \neq 0$ and $\nu = \nu_{\text{sht}}$. Recall that the latter constraint means that we calculate the order with respect to t_{sht} .

As another example, let us consider $w = w_0$. One needs to verify that

$$m_i(\rho_\nu^\vee, \alpha^i) \leq |\{ \tilde{\beta}^r ; \beta^r = s^i(\alpha^r) < 0, \nu_{\alpha^r} = \nu = \nu_{\beta^r}, 1 \leq r < i \}|$$

in this case. The right-hand side for $1 \leq r \leq i$, i.e. with i added to the range, has been already calculated above. So the required cardinality is $m_i(\rho_\nu^\vee, \alpha^i) - \delta_{\alpha^i, \nu}$ plus $(\rho_\nu^\vee, \alpha^i)$ for short α^i and $(\rho_\nu, (\alpha^i)^\vee)$ for long α^i . Thus the inequality holds and is strict unless α^i is a simple root and $\nu = \nu_i$.

4.5. One g and any w . This case is actually the key for the general consideration, which will be managed by induction with respect to the number of terms $\tilde{g}_{\tilde{\alpha}^i} \sigma^i$ taken in the products.

We will use that $\tilde{\beta}^r = \sigma^i(\tilde{\alpha}^i) \notin \{\tilde{\alpha}^i, \dots, \tilde{\alpha}^1\}$ for $r < i$ if and only if $\tilde{\beta}^r < 0$. Indeed,

$$\tilde{\beta}^r = \tilde{\alpha}^r - 2 \frac{(\alpha^i, \alpha^r)}{(\alpha^i, \alpha^i)} \tilde{\alpha}^i$$

and if $\tilde{\beta}^r > 0$, then either $\tilde{\beta}^r$ is a sum of two roots from $\lambda_\nu(b)$ with positive coefficients and therefore belongs to $\{\tilde{\alpha}^i, \dots, \tilde{\alpha}^1\}$ or $(\alpha^i, \alpha^r) > 0$ and $\tilde{\alpha}^r$ is such a sum of $\tilde{\beta}^r$ and $\tilde{\alpha}^i$. In the latter case, β^r must belong to $\lambda_\nu(b)$ because otherwise $\tilde{\alpha}^i$ would occur before $\tilde{\alpha}^r$ in this sequence; see Claim (d') Theorem 2.1 from [C9]. Moreover, $\tilde{\beta}^r$ must appear before $\tilde{\alpha}^i$ (actually before $\tilde{\alpha}^r$) in this case. This proves that $\tilde{\beta}^r < 0$.

For arbitrary $w \in W$ and $\nu \in \nu_R$, let us first consider $\tilde{\beta}^r$ such that

$$(4.24) \quad \begin{aligned} \text{ord}_w^\nu(\tilde{f}_{\tilde{\beta}^r}) &< \text{ord}_w^\nu(\tilde{f}_{\tilde{\alpha}^r}) \quad \text{for } r < i, \text{ equivalently,} \\ w^{-1}(\beta^r) &= w^{-1}(s_{\alpha^i}(\alpha^r)) < 0 \quad \text{and } w^{-1}(\alpha^r) > 0. \end{aligned}$$

We assume here and below that $\nu_{\alpha^r} = \nu = \nu_{\beta^r}$; otherwise we have $0 = 0$ in (4.24). Setting $\epsilon' = \text{sgn}(w^{-1}(\alpha^i))$ and $\alpha' \stackrel{\text{def}}{=} \epsilon' w^{-1}(\alpha^i) \in R_+$, we have

$$w^{-1}(\alpha^r) \in \lambda_\nu(s_{\alpha'}) \quad \text{and therefore } (w^{-1}(\alpha^r), \alpha') = \epsilon'(\alpha^r, \alpha^i) > 0.$$

We can stick here only to negative $\tilde{\beta}^r$ due to the remark above; only such $\tilde{\beta}^r$ may influence the difference $\sum_{r=1}^i \text{ord}_w^\nu(\tilde{f}_{\tilde{\beta}^r}) - \sum_{r=1}^i \text{ord}_w^\nu(\tilde{f}_{\tilde{\alpha}^r})$.

(a) *The case of $w^{-1}(\alpha^i) < 0$.* Accordingly, $\epsilon' = -$ and $(\alpha^r, \alpha^i) < 0$ for $\tilde{\alpha}^r$ under consideration. However then the corresponding $\tilde{\beta}^r =$

$\tilde{\alpha}^r - 2 \frac{(\alpha^i, \alpha^r)}{(\alpha^i, \alpha^i)} \tilde{\alpha}^i$ belong to $\lambda_\nu(b)$ and, moreover, coincides with certain $\tilde{\alpha}^p$ for $p < i$, as it was noted above. We see that

$$(4.25) \quad \sum_{r=1}^i \text{ord}_w^\nu(\tilde{f}_{\tilde{\beta}^r}) - \sum_{r=1}^i \text{ord}_w^\nu(\tilde{f}_{\tilde{\alpha}^r}) \geq 0.$$

If $m_i = 0$, i.e. α^i is nonaffine, this concludes the verification of (4.22) in the considered case. Otherwise, the difference from (4.25) must be large enough to compensate the negative t_ν -order due to $\tilde{g}_{\tilde{\alpha}^i} \sigma^i$. Let us address this.

We can now assume that $m_i \geq 1$. It suffices to consider $\tilde{\alpha}^r = [\alpha^r, \nu_{\alpha^r} m_r]$ with $\nu_{\alpha^r} m_r > 0$ and $\tilde{\alpha}^r = [\alpha^r, 0]$ with $\alpha^r \notin \lambda(s^i)$. The roots $\tilde{\alpha}^r$ from $\lambda(s^i)$ have been already used; recall that $s^i = s_{\alpha^i}$. Let us evaluate the corresponding number of pairs $\{\tilde{\alpha}^r, \tilde{\beta}^r < 0\}$ such that $\nu_{\alpha^r} = \nu = \nu_{\beta^r}$ and

$$(4.26) \quad \begin{aligned} \text{ord}_w^\nu(\tilde{f}_{\tilde{\beta}^r}) &> \text{ord}_w^\nu(\tilde{f}_{\tilde{\alpha}^r}) \quad \text{for } r < i, \text{ equivalently,} \\ w^{-1}(\beta^r) &= w^{-1}(s_{\alpha^i}(\alpha^r)) > 0 \quad \text{and } w^{-1}(\alpha^r) < 0. \end{aligned}$$

Thus $\alpha^i, \alpha^r \in \lambda_\nu(w^{-1})$. We continue to assume that $\epsilon' = -$; so $\alpha' = -w^{-1}(\alpha^i) > 0$.

Let us take an arbitrary $\gamma \in \lambda_\nu(s_{\alpha'})$; then $\gamma' = -s_{\alpha'}(\gamma)$ belongs to $\lambda_\nu(s_{\alpha'})$ as well. We will first consider the *ADE*-case. For an arbitrary j such that $m_i > j > 0$, we claim that either $[-w(\gamma), j]$ or $[-w(\gamma'), m_i - j]$ belongs to the sequence $\{\tilde{\alpha}^r, 1 \leq r \leq i\}$. Indeed,

$$(4.27) \quad \begin{aligned} w(-\gamma) + w(-\gamma') &= w(-\alpha') = \alpha^i \quad \text{and} \\ [w(-\gamma), j] + [w(-\gamma'), m_i - j] &= [w(-\alpha'), m_i] = \tilde{\alpha}^i. \end{aligned}$$

Therefore exactly one of these two roots must occur in the sequence $\lambda(b)$ before $\tilde{\alpha}^i$. Generally speaking this one can be with a negative nonaffine component, but all nonaffine components are positive in $\lambda(b)$ since $b \in P_+$.

Moreover, we can assume that $-w(\gamma) > 0$, since at least one of $w(-\gamma)$ and $w(-\gamma')$ must be positive. Then the claim is that either $w(-\gamma) = \tilde{\alpha}^r \notin \lambda(s^i)$ for certain $r < i$ or $[w(-\gamma'), m_i] = \tilde{\alpha}^r$ for $r \leq i$, which readily follows from (4.27) with $j = 0$.

We obtain that the number of $\tilde{\alpha}^r \notin \lambda(s^i)$ in the form $[-w(\gamma), j]$ for $\gamma \in \lambda(s_{\alpha'})$ will be $m_i((\rho, \alpha') - 1) + m_i = m_i(\rho, \alpha')$, where the second m_i counts the roots $[\alpha^i, j]$ for $1 \leq j \leq m_i$.

In the $BCFG$ -case, the calculation is very similar to that for (4.23); the general answer for $\tilde{\alpha}^r \notin \lambda(s^i)$ satisfying (4.26) is $m_i(\rho_\nu^\vee, \alpha')$. Indeed, for any root system R , we need to count the number \mathfrak{N}_ν of $\tilde{\gamma} = [\gamma, \nu_\gamma k]$ such that

$$\gamma \in \lambda_\nu(s_{\alpha'}) \setminus \{\alpha'\}, \quad 0 < \nu_\gamma k < \nu_{\alpha'} m_i, \quad [w(\gamma), \nu_\gamma k] \in \{\tilde{\alpha}^r, 1 \leq r \leq i\};$$

the consideration of the roots $\tilde{\alpha}^r \notin \lambda(s^i)$, with $k = 0$ and those for $\nu_\gamma k = \nu_{\alpha'} m_i$, is left to the readers. We obtain that

$$\mathfrak{N}_\nu = (m_i - 1)((\rho_\nu^\vee, \alpha) - \delta_{\alpha, \nu}) \quad \text{for long or short } \alpha = \alpha'.$$

The remaining $\tilde{\alpha}^r$ with $k = 0$ and those for $\nu_\gamma k = \nu_{\alpha'} m_i$ will change $(m_i - 1)$ in \mathfrak{N}_ν by m_i . Then we add the roots with $\gamma = \alpha'$ if $\nu_\alpha = \nu$ and obtain the required $m_i(\rho_\nu^\vee, \alpha')$.

To finalize this calculation we use (4.14):

$$\dagger \text{ord}_w^\nu(\mathfrak{a}^\delta(\tilde{g}_{\tilde{\alpha}^i} s_{\tilde{\alpha}^i})) = \dagger \text{ord}_w^\nu(\mathfrak{a}^\delta(s_{\alpha^i} \Gamma_{\alpha^i}^{-m_i} \tilde{g}_{-\tilde{\alpha}^i})) = m_i(\rho_\nu^\vee, w^{-1}(\alpha^i));$$

recall that $w^{-1}(\alpha^i) < 0$. Therefore (4.22) holds in this case.

(b) *The case of $w^{-1}(\alpha^i) > 0$.* Now $\epsilon' = +$ and $(\alpha^r, \alpha^i) > 0$. We fix $\nu \in \nu_R$. We can essentially follow the same verification as for (4.26) with $\tilde{w} = w_0 w$ instead of w . Indeed, $\tilde{w}^{-1}(\alpha^i) < 0$ and we need to count the number of pairs $\{\tilde{\alpha}^r, \tilde{\beta}^r \stackrel{\text{def}}{=} \sigma^i(\tilde{\alpha}^r)\}$ satisfying the same positivity conditions as in (4.26), namely, $\tilde{w}^{-1}(\alpha^r) < 0$ and $\tilde{w}^{-1}(\beta^r) > 0$.

However now the switch from $\text{ord}_w^\nu(\tilde{f}_{\tilde{\beta}^r})$ to $\text{ord}_w^\nu(\tilde{f}_{\tilde{\alpha}^r})$ in the corresponding pair $\{\tilde{\alpha}^r, \tilde{\beta}^r\}$ will increase

$$(4.28) \quad \text{ord}_w^\nu(\tilde{f}_{\tilde{\beta}^1} \tilde{f}_{\tilde{\beta}^2} \cdots \tilde{f}_{\tilde{\beta}^{i-1}}) = \text{ord}_w^\nu(\tilde{f}_{\tilde{\beta}^1}) + \dots + \text{ord}_w^\nu(\tilde{f}_{\tilde{\beta}^{i-1}})$$

from (4.22) by one. Accordingly, it suffices to know the upper bound for the number of such pairs, not the lower bound (actually, the exact number) needed in (4.26). Relation (4.22), which compares the sum in (4.28) plus $\dagger \text{ord}(\tilde{g}_{\tilde{\alpha}^i} \sigma^i)$ with that for $\tilde{\alpha}^r$ ($1 \leq r \leq i$), holds (only) due to the positive t_ν -orders of $\tilde{g}_{\tilde{\alpha}^i}$ and $\sigma^i = s_{\tilde{\alpha}^i}$.

In contrast to (4.26), we now have to include nonaffine $\tilde{\alpha}^r$ from $\lambda(s^i)$. This is straight and will be considered below. We obtain that the number of pairs $\{\tilde{\alpha}^r, \tilde{\beta}^r\}$ such that the substitution $\tilde{\beta}^r \mapsto \tilde{\alpha}^r$ increases (4.28) can be no greater than

$$(m_i + 1)((\rho_\nu^\vee, \alpha') - \delta_{\alpha', \nu}) + \delta_{\alpha', \nu} m_i = (m_i + 1)(\rho_\nu^\vee, \alpha') - \delta_{\alpha', \nu};$$

recall that $\alpha' = w^{-1}(\alpha^i)$ and $\nu \in \nu_R$ is fixed.

On the other hand, (4.14) results in

$${}^{\dagger}\mathrm{ord}_w^{\nu}(\mathfrak{ae}^{\delta}(\tilde{g}_{\tilde{\alpha}^i} s_{\tilde{\alpha}^i})) = (m_i + 1) (\rho_{\nu}^{\vee}, w^{-1}(\alpha^i)).$$

Thus the change of this order versus $\mathrm{ord}_w^{\nu}(\mathfrak{ae}^{\delta}(\tilde{f}_{\tilde{\alpha}^i})) = \delta_{\alpha', \nu}$ is exactly $(m_i + 1)(\rho_{\nu}^{\vee}, \alpha') - \delta_{\alpha', \nu}$ and therefore it “compensates” the total sum of negative differences $\mathrm{ord}_w^{\nu}(\tilde{f}_{\tilde{\beta}^r} - \tilde{f}_{\tilde{\alpha}^r})$ for $r < i$. We establish that the inequality from (4.22) holds when $w^{-1}(\alpha^i) > 0$ and conclude the justification of Lemma 4.24.

5. TODA-DUNKL OPERATORS

The general case will be managed by induction with respect to the number of s^i taken in the products. We will begin with some notations and basic estimates.

Recall that for $\tilde{\alpha} = [\alpha, \nu_{\alpha} j] \in \tilde{R}$, $t_{\tilde{\alpha}} = t_{\alpha} = t_{\nu_{\alpha}}$,

$$\begin{aligned} G_{\tilde{\alpha}}^{+} &\stackrel{\text{def}}{=} t_{\alpha}^{1/2} + \frac{t_{\alpha}^{1/2} - t_{\alpha}^{-1/2}}{X_{\tilde{\alpha}}^{-1} - 1} (1 - s_{\tilde{\alpha}}) = t_{\alpha}^{-1/2} (f_{\tilde{\alpha}} + g_{\tilde{\alpha}} s_{\tilde{\alpha}}), \\ f_{\tilde{\alpha}} &= \frac{t_{\alpha} X_{\tilde{\alpha}}^{-1} - 1}{X_{\tilde{\alpha}}^{-1} - 1}, \quad g_{\tilde{\alpha}} = \frac{t_{\alpha} - 1}{1 - X_{\tilde{\alpha}}^{-1}}; \\ G_{\tilde{\alpha}}^{-} &\stackrel{\text{def}}{=} t_{\alpha}^{-1/2} + \frac{t_{\alpha}^{1/2} - t_{\alpha}^{-1/2}}{1 - X_{\tilde{\alpha}}^{-1}} (1 - s_{\tilde{\alpha}}) = t_{\alpha}^{-1/2} (f_{\tilde{\alpha}} - s_{\tilde{\alpha}} g_{\tilde{\alpha}}). \end{aligned}$$

Also, $\ddot{G}_{\tilde{\alpha}}^{\pm} \stackrel{\text{def}}{=} t_{\tilde{\alpha}}^{1/2} G_{\tilde{\alpha}}^{\pm}$ and $Y_b = q^{(b-, \rho_k)} b \ddot{G}_{\tilde{\alpha}^l}^{\mathrm{sgn}(\epsilon_l)} \cdots \ddot{G}_{\tilde{\alpha}^1}^{\mathrm{sgn}(\epsilon_1)}$ for $b \in P$; see (2.6). Note that $\ddot{G}_{-\tilde{\alpha}}^{-} = \tilde{G}_{\tilde{\alpha}}^{-}$ was used in (4.6).

See the definition and basic properties of ord_w^{ν} and ${}^{\dagger}\mathrm{ord}_w^{\nu}$ in the beginning of Section 4.1.

The following proposition provides the basic tool needed for a direct proof of the existence of the Toda-Dunkl operators. The proposition consists of two parts. In part A, given $u \in W$ and a reduced decomposition $u = s_{j_l} \cdots s_{j_1}$, we will consider arbitrary products of the form

$$(5.1) \quad \ddot{G}_{\alpha^p}^{\pm} \cdots \ddot{G}_{\alpha^r}^{\pm}, \quad \ddot{G}_{-\alpha^r}^{\pm} \cdots \ddot{G}_{-\alpha^p}^{\pm} \quad \text{for } 1 \leq r \leq p \leq l,$$

and expand such products by choosing from each \ddot{G}_{α}^{\pm} either f_{α} or $g_{\alpha} s_{\alpha}$. The statements in part B are more restrictive, though they directly result in the existence of the Toda-Dunkl operators; see Corollary 5.2. We will see in Proposition 5.4 that it is sufficient to manage only the

nonaffine products from (5.1) in order to prove the existence of the Toda-Dunkl operators.

5.1. The key step. The following proposition is the key in our approach.

Proposition 5.1. *A. For $u \in W$ and its reduced decomposition $u = s_{j_l} \cdots s_{j_1}$, let $1 \leq r \leq p \leq l$ and $w \in W$.*

(i) The ord_w^ν of any product in the expansion of $\mathfrak{ae}^\delta(\ddot{G}_{\alpha^p}^+ \cdots \ddot{G}_{\alpha^r}^+)$ is bounded below by $\text{ord}_w^\nu(\mathfrak{ae}^\delta(f_{\alpha^p} \cdots f_{\alpha^r}))$.

(ii) The ord_w^ν of any product in the expansion of $\mathfrak{ae}^\delta(\ddot{G}_{-\alpha^r}^+ \cdots \ddot{G}_{-\alpha^p}^+)$ is bounded below by $\text{ord}_w^\nu(\mathfrak{ae}^\delta(f_{-\alpha^r} \cdots f_{-\alpha^p}))$.

(iii) The ${}^\dagger\text{ord}_w^\nu$ of any product in the expansion of $\mathfrak{ae}^\delta(\ddot{G}_{\alpha^p}^- \cdots \ddot{G}_{\alpha^r}^-)$ is bounded below by $\text{ord}_w^\nu(\mathfrak{ae}^\delta(f_{\alpha^p} \cdots f_{\alpha^r}))$.

(iv) The ${}^\dagger\text{ord}_w^\nu$ of any product in the expansion of $\mathfrak{ae}^\delta(\ddot{G}_{-\alpha^r}^- \cdots \ddot{G}_{-\alpha^p}^-)$ is bounded below by $\text{ord}_w^\nu(\mathfrak{ae}^\delta(f_{-\alpha^r} \cdots f_{-\alpha^p}))$.

B. Let $b = \pi_r s_{j_l} \cdots s_{j_1}$ be a reduced decomposition of $b \in P_+$ and let $1 \leq p \leq l$, $w \in W$.

(i) The ord_w^ν of any product in the expansion of $\mathfrak{ae}^\delta(\ddot{G}_{\tilde{\alpha}^p}^+ \cdots \ddot{G}_{\tilde{\alpha}^1}^+)$ is bounded below by $\text{ord}_w^\nu(\mathfrak{ae}^\delta(f_{\tilde{\alpha}^p} \cdots f_{\tilde{\alpha}^1}))$.

(ii) The ${}^\dagger\text{ord}_w^\nu$ of any product in the expansion of $\mathfrak{ae}^\delta(\ddot{G}_{-\tilde{\alpha}^1}^- \cdots \ddot{G}_{-\tilde{\alpha}^p}^-)$ is bounded below by $\text{ord}_w^\nu(\mathfrak{ae}^\delta(f_{-\alpha^1} \cdots f_{-\alpha^p}))$.

Corollary 5.2. *Let $b \in P_+$. The ord_w^ν of any particular product in the expansions of*

$$\mathfrak{ae}^\delta(b^{-1}Y_b) = G_{\tilde{\alpha}^l}^+ \cdots G_{\tilde{\alpha}^1}^+ \quad \text{or} \quad \mathfrak{ae}^\delta(Y_b^{-1}b) = G_{\tilde{\alpha}^1}^- \cdots G_{\tilde{\alpha}^l}^-$$

is no smaller than $\text{ord}_w^\nu(b)$ and $\text{ord}_w^\nu(b^{-1})$ correspondingly. Therefore, the operators $\hat{Y}_b = RE^\delta(Y_b)$ and $\hat{Y}_b' = RE^\delta(Y_b^{-1})$ are well defined and invertible; $Y_b Y_b^{-1} = 1$ results in $\hat{Y}_b \hat{Y}_b' = 1$. Moreover, the RE^δ -limits of all products in their G -expansions are well defined.

The corollary readily follows from Part B of the proposition. For instance, use (ii) and calculations performed in (4.16) and (4.17) in the case of Y_b^{-1} .

We provide below a complete proof only for Claim (i) from Part A of the proposition. Statements (ii, iii, iv) and Part B can be proved by similar arguments. The justification is based on the following lemma.

Lemma 5.3. *A. Let $u \in W$, choose a reduced decomposition $u = s_{j_l} \cdots s_{j_1}$, and let $1 \leq r \leq p \leq l$. Then for any $p \geq i \geq r$ and $w \in W$, one has*

$$(5.2) \quad \text{ord}_w^\nu(\mathfrak{ae}^\delta(g_{\alpha^i} f_{s^i(\alpha^{i-1})} \cdots f_{s^i(\alpha^r)})) \geq \text{ord}_w^\nu(\mathfrak{ae}^\delta(f_{\alpha^i} \cdots f_{\alpha^r})).$$

B. We use the notation $\sigma^k = s_{\tilde{\alpha}^k}$, where $1 \leq k \leq l$. For a reduced decomposition $b = \pi_r s_{j_l} \cdots s_{j_1} \in P_+$ and any $1 \leq i \leq l$, $w \in W$,

$$(5.3) \quad \text{ord}_w^\nu(\mathfrak{ae}^\delta(g_{\tilde{\alpha}^i} f_{\sigma^i(\tilde{\alpha}^{i-1})} \cdots f_{\sigma^i(\tilde{\alpha}^1)} \sigma^i)) \geq \text{ord}_w^\nu(\mathfrak{ae}^\delta(f_{\tilde{\alpha}^i} \cdots f_{\tilde{\alpha}^1})).$$

5.2. The justifications.

Proof of Lemma 5.3. We prove Part A only; the second part is quite parallel to Section 4.5 instead of the arguments below. Actually, formula (4.26) above is the only really special feature of the affine case. This formula and related ones were given in Section 4.5 in the case of Y_b^{-1} ; the adjustments needed for (5.3), the case of Y_b , are straightforward. This formula is exactly the reason why we need to make $r = 1$ when extending (5.2) to the affine case.

Let us begin with some basic orders. We note first that for $\tilde{\alpha} = [\alpha, \nu_\alpha j]$

$$(5.4) \quad \text{ord}_w^\nu(\mathfrak{ae}^\delta(g_{\tilde{\alpha}})) = \begin{cases} 0, & \text{if } w^{-1}(\alpha) > 0, \\ -(\rho_\nu^\vee, w^{-1}(\alpha)), & \text{if } w^{-1}(\alpha) < 0, \end{cases}$$

and

$$(5.5) \quad \text{ord}_w^\nu(\mathfrak{ae}^\delta(f_{\tilde{\alpha}})) = \begin{cases} 0, & \text{if } w^{-1}(\alpha) > 0, \\ \delta_{\nu, \nu_\alpha}, & \text{if } w^{-1}(\alpha) < 0. \end{cases}$$

The second line in (5.5) follows from the fact that $(\rho_\nu^\vee, w^{-1}(\alpha)) \neq 0$ for all $w \in W$ provided $\nu = \nu_\alpha$.

Let $u = s_{j_l} \cdots s_{j_1}$ be the reduced decomposition from Part A and $r \leq i \leq p \leq l$. Write $\alpha = \alpha^i$ (so $s_\alpha = s^i$) and take $\beta = \alpha^k$ for any $i > k \geq r$. Using (5.5), one has

$$\text{ord}_w^\nu(\mathfrak{ae}^\delta(f_\beta)) \leq \text{ord}_w^\nu(\mathfrak{ae}^\delta(f_{s_\alpha(\beta)}))$$

unless

$$(5.6) \quad \nu = \nu_\beta, \quad w^{-1}(\beta) < 0, \quad \text{and} \quad w^{-1}(s_\alpha(\beta)) > 0.$$

An equivalent description of (5.6) is

$$(5.7) \quad \text{ord}_w^\nu(\mathfrak{a}^\delta(f_\beta)) = 1 \quad \text{and} \quad \text{ord}_w^\nu(\mathfrak{a}^\delta(f_{s_\alpha(\beta)})) = 0,$$

Assuming (5.6) holds, there are two cases to consider: either $w^{-1}(\alpha) > 0$ or $w^{-1}(\alpha) < 0$.

Suppose $w^{-1}(\alpha) > 0$. Then $\text{ord}_w^\nu(\mathfrak{a}^\delta(g_\alpha)) = \text{ord}_w^\nu(\mathfrak{a}^\delta(f_\alpha)) = 0$. If (5.7) occurs, then one must have $(\beta, \alpha) < 0$. Hence $s_\alpha(\beta)$ belongs to $\lambda(u)$ and by Lemma 1.1, one has $s_\alpha(\beta) = \alpha^j$ where $i > j > k$. Therefore, the application of s_α to the product $f_{\alpha^{i-1}} \cdots f_{\alpha^r}$ reverses the positions of the factors $f_{s_\alpha(\beta)}$ and f_β for all pairs $\{\beta, s_\alpha(\beta)\}$, where β satisfies (5.7); the ord_w^ν of any other factors in this product can only increase upon the application of s_α . This proves (5.2) when $w^{-1}(\alpha) > 0$.

It remains to consider the case when $w^{-1}(\alpha) < 0$. We note that $w^{-1}(s_\alpha(\beta)) = s_{w^{-1}(\alpha)}(w^{-1}(\beta))$. By (1.28), one has

$$(5.8) \quad l_\nu(s_{w^{-1}(\alpha)}) \leq -2(\rho_\nu^\vee, w^{-1}(\alpha)) - \delta_{\nu, \nu_\alpha}.$$

(The only case when (5.8) is not an equality is $\nu_\alpha = \nu_{\text{lng}}$ and $\nu = \nu_{\text{sht}}$.) Combining this with (5.4) and (5.5) yields

$$(5.9) \quad \text{ord}_w^\nu(\mathfrak{a}^\delta(g_\alpha)) \geq \text{ord}_w^\nu(\mathfrak{a}^\delta(f_\alpha)) + \frac{l_\nu(s_{w^{-1}(\alpha)}) - \delta_{\nu, \nu_\alpha}}{2}.$$

Using (1.28), one sees that

$$\frac{l_\nu(s_{w^{-1}(\alpha)}) - \delta_{\nu, \nu_\alpha}}{2}$$

is the maximum possible number of β satisfying (5.6). In other words, (5.9) compensates for all drops in the order coming from (5.7) when applying s_α to the product $f_{\alpha^{i-1}} \cdots f_{\alpha^r}$. This establishes (5.2). \square

Now we are going to prove Statement (i) from Part A of Proposition 5.1. We argue by induction on the number of factors of the form $g_\alpha s_\alpha$ chosen to form a particular product in the expansion — the base case being the product when no such factors are chosen, i.e., $\mathcal{P}^\emptyset \stackrel{\text{def}}{=} \mathfrak{a}^\delta(f_{\alpha^p} \cdots f_{\alpha^r})$.

First, let us consider some particular cases. Suppose that just one factor of the form $g_\alpha s_\alpha$, say $g_{\alpha^i} s^i$, is chosen. In other words, take the product

$$\mathcal{P}^i \stackrel{\text{def}}{=} \mathfrak{a}^\delta(f_{\alpha^p} \cdots f_{\alpha^{i+1}} g_{\alpha^i} s^i f_{\alpha^{i-1}} \cdots f_{\alpha^r}).$$

Due to (4.3),

$$\begin{aligned} \text{ord}_w^\nu(\mathcal{P}^i) &= \text{ord}_w^\nu(\mathfrak{ae}^\delta(f_{\alpha^p} \cdots f_{\alpha^{i+1}} g_{\alpha^i} f_{s^i(\alpha^{i-1})} \cdots f_{s^i(\alpha^r)})) \\ &= \text{ord}_w^\nu(\mathfrak{ae}^\delta(f_{\alpha^p} \cdots f_{\alpha^{i+1}})) + \text{ord}_w^\nu(\mathfrak{ae}^\delta(g_{\alpha^i} f_{s_{\alpha^i}(\alpha^{i-1})} \cdots f_{s_{\alpha^i}(\alpha^r)})), \end{aligned}$$

Then (5.2) gives $\text{ord}_w^\nu(\mathcal{P}^i) \geq \text{ord}_w^\nu(\mathcal{P}^\emptyset)$, as claimed.

Next, let us consider the case when two factors of $g_\alpha s_\alpha$ are chosen:

$$\mathcal{P}^{ij} \stackrel{\text{def}}{=} \mathfrak{ae}^\delta(f_{\alpha^p} \cdots f_{\alpha^{i+1}} (g_{\alpha^i} s^i) f_{\alpha^{i-1}} \cdots f_{\alpha^{j+1}} (g_{\alpha^j} s^j) f_{\alpha^{j-1}} \cdots f_{\alpha^r}).$$

Due to (4.3),

$$\begin{aligned} \text{ord}_w^\nu(\mathcal{P}^{ij}) &= \text{ord}_w^\nu(\mathfrak{ae}^\delta(f_{\alpha^p} \cdots f_{\alpha^{i+1}} g_{\alpha^i} f_{s^i(\alpha^{i-1})} \cdots f_{s^i(\alpha^{j+1})} \\ (5.10) \quad &\quad \times g_{s^i(\alpha^j)} f_{s^i s^j(\alpha^{j-1})} \cdots f_{s^i s^j(\alpha^r)})). \end{aligned}$$

Apply (4.2) and (5.2) as follows:

$$\begin{aligned} &\text{ord}_w^\nu(\mathfrak{ae}^\delta(g_{s^i(\alpha^j)} f_{s^i s^j(\alpha^{j-1})} \cdots f_{s^i s^j(\alpha^r)})) \\ &= \text{ord}_{s^i w}^\nu(\mathfrak{ae}^\delta(g_{\alpha^j} f_{s^j(\alpha^{j-1})} \cdots f_{s^j(\alpha^r)})) \\ &\geq \text{ord}_{s^i w}^\nu(\mathfrak{ae}^\delta(f_{\alpha^j} f_{\alpha^{j-1}} \cdots f_{\alpha^r})) = \text{ord}_w^\nu(\mathfrak{ae}^\delta(f_{s^i(\alpha^j)} f_{s^i(\alpha^{j-1})} \cdots f_{s^i(\alpha^r)})). \end{aligned}$$

Returning to (5.10), one then has

$$\begin{aligned} (5.11) \quad \text{ord}_w^\nu(\mathcal{P}^{ij}) &\geq \text{ord}_w^\nu(\mathfrak{ae}^\delta(f_{\alpha^p} \cdots f_{\alpha^{i+1}} g_{\alpha^i} f_{s^i(\alpha^{i-1})} \cdots f_{s^i(\alpha^r)})) \\ &= \text{ord}_w^\nu(\mathcal{P}^i) \geq \text{ord}_w^\nu(\mathcal{P}^\emptyset). \end{aligned}$$

In general, for any decreasing sequence $p \geq i_1 > i_2 > \cdots > i_m \geq r$, we set

$$\mathcal{P}^{i_1 \dots i_m} \stackrel{\text{def}}{=} \mathfrak{ae}^\delta(h_p \cdots h_r),$$

where $h_i = g_{\alpha^i} s^i$ whenever $i \in \{i_1, \dots, i_m\}$ and $h_i = f_{\alpha^i}$ otherwise. The same reasoning used to arrive at (5.11) shows that

$$(5.12) \quad \text{ord}_w^\nu(\mathcal{P}^{i_1 \dots i_m}) \geq \text{ord}_w^\nu(\mathcal{P}^{i_1 \dots i_{m-1}}),$$

which gives the induction step and completes the proof. \square

Omitting all f_α . As an example clarifying the nature of the estimates in Lemma 5.3, let us discuss the extremal case of Proposition 5.1 when we choose all $g_\alpha s_\alpha$ when expanding $\mathfrak{ae}^\delta(\ddot{G}_{\alpha^l}^+ \cdots \ddot{G}_{\alpha^1}^+)$. That is, consider the product

$$\mathfrak{ae}^\delta(g_{\alpha^l} s^l \cdots g_{\alpha^1} s^1) = \mathfrak{ae}^\delta(g_{-u^{-1}(\alpha_{j_l})} \cdots g_{-u^{-1}(\alpha_{j_1})} u^{-1}).$$

Let $M_j^\nu(u; w)$ denote the number of simple reflections s_j for $\nu_j = \nu$ in the given decomposition $u = s_{j_l} \cdots s_{j_1}$ such that $(uw)^{-1}(\alpha_j) > 0$ (so it

depends on the choice of the reduced decomposition). Using (5.4) and (5.5), Proposition 5.1(i) then translates to

$$\sum_{j=1}^n M_j^\nu(u; w) (\alpha_j, uw(\rho_\nu^\vee)) \geq \sum_{\alpha \in \lambda(u) \cap \lambda(w^{-1})} \delta_{\nu, \alpha}.$$

Similarly, Proposition 5.1(iii) in such a case leads to the product

$$\mathfrak{ae}^\delta(s^l g_{\alpha^l} \cdots s^1 g_{\alpha^1}) = \mathfrak{ae}^\delta(u^{-1} g_{\alpha_{j_l}} \cdots g_{\alpha_{j_1}})$$

and gives that

$$(5.13) \quad - \sum_{j=1}^n \delta_{\nu, \alpha_j} N_j(u; w) (\alpha_j, w(\rho_\nu^\vee)) \geq \sum_{\alpha \in \lambda(u) \cap \lambda(w^{-1})} \delta_{\nu, \alpha},$$

where $N_j(u; w)$ is the number of simple reflections s_j in $u = s_{j_l} \cdots s_{j_1}$ such that $w^{-1}(\alpha_j) < 0$. It is a counterpart of formula (4.19) with $u \in W$ instead of $b \in P_+$.

When $w = w_0$, relation (5.13) becomes the identity $l_\nu(u) = l_\nu(u)$. Taking $w = u^{-1}$ and assuming that there is only one simple α_j in $\lambda_\nu(u)$, we obtain the following upper bound for $l_\nu(u)$:

$$-N_j(u)(u(\alpha_j), \rho_\nu^\vee) \geq l_\nu(u), \quad N_j(u) = |\{j_r = j, r = 1, 2, \dots, l\}|.$$

5.3. Y -hat operators. In this section, we use the considerations above to give a direct and constructive justification of the existence of the Toda-Dunkl operators, based on the consideration of minuscule weights, ϑ and short roots (considered as weights).

Proposition 5.4. *The limit $RE^\delta(Y_b)$ exists when*

- (1) $b = \omega_r$ ($r \in O'$),
- (2) b is a short positive root,
- (3) $b = -\omega_r$ ($r \in O'$),
- (4) b is a short negative root,

and therefore it exists for any $b \in P$.

Proof. The final claim holds because P is generated by the minuscule weights together with Q , which in turn is generated by the short roots.

For (1) and (2), we consider first $b \in P_+$. Write $b = \pi_r \tilde{w} = \pi_r s_{j_l} \cdots s_{j_1}$ ($l = l(b)$) in \widehat{W} and form $\tilde{\alpha}^p$ ($1 \leq p \leq l$) from (1.15). Since $b \in P_+$, one has $l_\nu(b) = 2(b, \rho_\nu^\vee)$ and $Y_b = T_b = \pi_r T_{j_l} \cdots T_{j_1}$. Hence $Y_b = q^{-(b, \rho_k)} \Gamma_{-b} \ddot{G}_{\tilde{\alpha}^l}^+ \cdots \ddot{G}_{\tilde{\alpha}^1}^+$. Using (3.36), we can write

$$\mathfrak{ae}^\delta(Y_b) = \sum_{w \in W} q^{-(b, \rho_k - w(\rho_k))} \Gamma_{-w^{-1}(b)} \zeta_w \mathfrak{ae}^\delta(\ddot{G}_{\tilde{\alpha}^l}^+ \cdots \ddot{G}_{\tilde{\alpha}^1}^+).$$

We claim that for any $b \in P_+$,

$$(5.14) \quad \xi^\delta(Y_b) \stackrel{\text{def}}{=} \sum_{w \in W} q^{-(b, \rho_k - w(\rho_k))} \Gamma_{-w^{-1}(b)} \zeta_w \mathfrak{ae}^\delta(f_{\tilde{\alpha}^l} \cdots f_{\tilde{\alpha}^1})$$

is regular at $t_\nu = 0$. Indeed, one has

$$q^{-(b, \rho_k - w(\rho_k))} = \prod_{\nu} t_{\nu}^{-(b, \rho_{\nu}^{\vee} - w(\rho_{\nu}^{\vee}))}$$

and the exponents $(b, \rho_{\nu}^{\vee} - w(\rho_{\nu}^{\vee}))$ count the number of $\tilde{\alpha} = [\alpha, \nu_{\alpha} j] \in \lambda_{\nu}(b)$ such that $w^{-1}(\alpha) < 0$. This follows from (1.16) and the following:

$$(5.15) \quad \rho_{\nu}^{\vee} - w(\rho_{\nu}^{\vee}) = \sum_{\alpha \in \lambda_{\nu}(w^{-1})} \alpha^{\vee}.$$

Now the regularity of $\xi^\delta(Y_b)$ is immediate from (5.5).

(1) Let $b = \omega_r$ for $r \in O'$; recall that $\omega_r = \pi_r u_r$. Using Proposition 5.1, where we take $u = u_r$, the regularity of $\mathfrak{ae}^\delta(Y_{\omega_r})$ follows from that of $\xi^\delta(Y_{\omega_r})$.

(2) Suppose $b = \alpha$ is any short positive root. Using Lemma 1.4, find a reduced expression $s_{\vartheta} = s_{j_1} \cdots s_{j_p} s_m s_{j_p} \cdots s_{j_1}$ such that $\alpha = s_{j_r} \cdots s_{j_1}(\vartheta)$ where $0 \leq r \leq p$. Let $l = l(s_{\vartheta}) = 2p + 1$ and construct $\lambda(s_{\alpha}) = \{\alpha^1, \dots, \alpha^l\}$ using the chosen reduced decomposition.

Recall that $\vartheta = s_0 s_{\vartheta}$ and $l(\vartheta) = l(s_{\vartheta}) + 1$. Accordingly, one has $Y_{\vartheta} = T_0 T_{s_{\vartheta}}$ and $\lambda(\vartheta) = \lambda(s_{\vartheta}) \cup \{[\vartheta, 1]\}$.

Due to (1.33) and (1.45), one has

$$Y_{\alpha} = (T_{j_r}^{-1} \cdots T_{j_1}^{-1}) T_0 (T_{j_1} \cdots T_{j_p} T_m T_{j_p} \cdots T_{j_{r+1}}).$$

Hence, for $v = s_{j_r} \cdots s_{j_1}$, we can write

$$(5.16) \quad v^{-1} Y_{\alpha} v = q^{-(\vartheta, \rho_k)} \ddot{G}_{\alpha^r}^- \cdots \ddot{G}_{\alpha^1}^- \Gamma_{-\vartheta} \ddot{G}_{[\vartheta, 1]}^+ \ddot{G}_{\alpha^l}^+ \cdots \ddot{G}_{\alpha^{r+1}}^+.$$

We note that by (3.34)

$$\mathfrak{ae}^\delta(v^{-1} Y_{\alpha} v) = v^{-1} \mathfrak{ae}^\delta(Y_{\alpha}) v.$$

Hence it suffices to prove that $\mathfrak{ae}^\delta(v^{-1} Y_{\alpha} v)$ is regular at $t_{\nu} = 0$.

By Proposition 5.1(i, iii), it is enough to consider

$$q^{-(\vartheta, \rho_k)} \mathfrak{ae}^\delta(f_{\alpha^r} \cdots f_{\alpha^1} \Gamma_{-\vartheta} \ddot{G}_{[\vartheta, 1]}^+ f_{\alpha^l} \cdots f_{\alpha^{r+1}}).$$

We expand this product by choosing either $f_{[\vartheta, 1]}$ or $g_{[\vartheta, 1]} s_{[\vartheta, 1]}$ from $\ddot{G}_{[\vartheta, 1]}^+$.

Choosing $f_{[\vartheta,1]}$ from $\ddot{G}_{[\vartheta,1]}^+$, we arrive at $\xi^\delta(Y_\vartheta)$, which is known to be regular at $t_\nu = 0$ (cf. (5.14)).

Thus it remains to choose $g_{[\vartheta,1]} s_{[\vartheta,1]}$. This yields

$$q^{-(\vartheta, \rho_k)} \mathfrak{a}^\delta(f_{\alpha^r} \cdots f_{\alpha^1} g_{[\vartheta,-1]} s_\vartheta f_{\alpha^l} \cdots f_{\alpha^{r+1}}),$$

where we have used that $\Gamma_{-\vartheta} g_{[\vartheta,1]} s_{[\vartheta,1]} = g_{[\vartheta,-1]} s_\vartheta$. According to (4.3), when calculating ord_w^ν , one must move s_ϑ to the right:

$$s_\vartheta (f_{\alpha^l} \cdots f_{\alpha^{r+1}}) = (f_{-\alpha^1} \cdots f_{-\alpha^{l-r}}) s_\vartheta,$$

where we have used (1.25). By (5.4), we need to show that

$$(5.17) \quad \begin{aligned} & \text{ord}_w^\nu(\mathfrak{a}^\delta(f_{\alpha^r} \cdots f_{\alpha^1} f_{-\alpha^1} \cdots f_{-\alpha^{l-r}})) \\ & \geq \begin{cases} (\vartheta, \rho_\nu^\vee), & \text{if } w^{-1}(\vartheta) > 0, \\ (\vartheta, \rho_\nu^\vee + w(\rho_\nu^\vee)), & \text{if } w^{-1}(\vartheta) < 0, \end{cases} \end{aligned}$$

for any $0 \leq r \leq p$.

To this end, assume first that $w^{-1}(\vartheta) > 0$. Clearly we have

$$\text{ord}_w^\nu(\mathfrak{a}^\delta(f_{\alpha^r} \cdots f_{\alpha^1} f_{-\alpha^1} \cdots f_{-\alpha^r})) = \sum_{i=1}^r \delta_{\nu, \alpha^i}.$$

For the remaining factors in the left-hand side of (5.17), one has

$$\text{ord}_w^\nu(\mathfrak{a}^\delta(f_{-\alpha^{r+1}} \cdots f_{-\alpha^{l-r}})) \geq \delta_{\nu, \vartheta} + \sum_{i=r+1}^p \delta_{\nu, \alpha^i}.$$

This can be seen as follows. First, $\alpha^{p+1} = \vartheta$ and hence by (5.5) $\text{ord}_w^\nu(\mathfrak{a}^\delta(f_{-\alpha^{p+1}})) = \delta_{\nu, \vartheta}$. Second, for each $r+1 \leq i \leq p$, at least one of $w^{-1}(\alpha^i)$ or $w^{-1}(\alpha^{l-i+1})$ must be positive. This follows from Lemma 1.1(ii), (1.25), and the assumption $w^{-1}(\vartheta) > 0$. Therefore, altogether one has

$$(5.18) \quad \text{ord}_w^\nu(\mathfrak{a}^\delta(f_{\alpha^r} \cdots f_{\alpha^1} f_{-\alpha^1} \cdots f_{-\alpha^{l-r}})) \geq \delta_{\nu, \vartheta} + \sum_{i=1}^p \delta_{\nu, \alpha^i}.$$

Finally, using Lemma 1.3(i), one finds that the right-hand side of (5.18) is exactly $(\vartheta, \rho_\nu^\vee)$.

Now assume $w^{-1}(\vartheta) < 0$. One has

$$(5.19) \quad \text{ord}_w^\nu(\mathfrak{a}^\delta(f_{-\alpha^1} \cdots f_{-\alpha^{l-r}})) = \sum_{\substack{1 \leq i \leq l-r \\ w^{-1}(\alpha^i) > 0}} \delta_{\nu, \alpha^i}$$

and

$$(5.20) \quad \text{ord}_w^\nu(\mathfrak{ae}^\delta(f_{\alpha^r} \cdots f_{\alpha^1})) = \sum_{\substack{1 \leq i \leq r \\ w^{-1}(\alpha^i) < 0}} \delta_{\nu, \alpha^i} \geq \sum_{\substack{l-r+1 \leq i \leq l \\ w^{-1}(\alpha^i) > 0}} \delta_{\nu, \alpha^i}.$$

The inequality in (5.20) follows from Lemma 1.1(ii), (1.25), and the assumption that $w^{-1}(\vartheta) < 0$. In particular, if $w^{-1}(\alpha^{l-i+1}) > 0$ for $1 \leq i \leq p$, then necessarily $w^{-1}(\alpha^i) < 0$. Putting (5.19) and (5.20) together, one has

$$(5.21) \quad \text{ord}_w^\nu(\mathfrak{ae}^\delta(f_{\alpha^r} \cdots f_{\alpha^1} f_{-\alpha^1} \cdots f_{-\alpha^{l-r}})) \geq \sum_{\substack{1 \leq i \leq l \\ w^{-1}(\alpha^i) > 0}} \delta_{\nu, \alpha^i}.$$

Finally, to get (5.17), we observe that

$$(5.22) \quad \rho_\nu^\vee + w(\rho_\nu^\vee) = \sum_{\substack{\alpha > 0, \nu_\alpha = \nu \\ w^{-1}(\alpha) > 0}} \alpha^\vee$$

and consequently $(\vartheta, \rho_\nu^\vee + w(\rho_\nu^\vee))$ is exactly the number of $\tilde{\alpha} \in \lambda_\nu(\vartheta)$ with $w^{-1}(\alpha) > 0$. Note that since $w^{-1}(\vartheta) < 0$, such $\tilde{\alpha}$ must belong to $\lambda_\nu(s_\vartheta) \setminus \{\vartheta\}$.

This completes the proof of (5.17) and hence the proof of (2) as well.

Comment. The relation $T_i^{-1} Y_b T_i^{-1} = Y_{s_i(b)}$ from (1.45), which was used at the beginning of (2), is valid only when $(b, \alpha_i^\vee) = 1$. In particular, it does not hold for $b = \alpha_m$ and $i = m$. In this case,

$$(5.23) \quad T_m^{-1} Y_{\alpha_m} T_m^{-1} = Y_{\alpha_m}^{-1} + (t_m^{1/2} - t_m^{-1/2}) T_m^{-1}.$$

One cannot use (5.23) to pass from Y_{α_m} to $Y_{\alpha_m}^{-1}$ in a way that is compatible with the limit $t_\nu \rightarrow 0$. Nevertheless, we can reach $Y_{\alpha_m}^{-1}$, along with all the operators corresponding to negative short roots, by starting from Y_ϑ^{-1} . This is carried out in (4) below. \square

Before (3) and (4), let us make some general remarks about Y_{-b} for arbitrary $b \in P_+$. Write $b = \pi_r s_{j_l} \cdots s_{j_1}$ ($l = l(b)$) and construct $\lambda(b) = \{\tilde{\alpha}^1, \dots, \tilde{\alpha}^l\}$ using this reduced decomposition. Then $-b = \pi_r^{-1} s_{\pi_r(j_1)} \cdots s_{\pi_r(j_l)}$, which is a reduced decomposition.

For $1 \leq p \leq l$, let

$$\tilde{\beta}^p = -b(\tilde{\alpha}^{l-p+1}) = s_{\pi_r(j_l)} \cdots s_{\pi_r(j_{l-p+2})}(\alpha_{\pi_r(j_{l-p+1})}),$$

so that $\lambda(-b) = \{\tilde{\beta}^1, \dots, \tilde{\beta}^l\}$. We can write

$$Y_{-b} = q^{-(b, \rho_k)} \Gamma_b \ddot{G}_{\tilde{\beta}^l}^- \cdots \ddot{G}_{\tilde{\beta}^1}^- = \ddot{G}_{-\tilde{\alpha}^1}^- \cdots \ddot{G}_{-\tilde{\alpha}^l}^- q^{-(b, \rho_k)} \Gamma_b.$$

Hence

$$(5.24) \quad \mathfrak{ae}^\delta(Y_{-b}) = \mathfrak{ae}^\delta(\ddot{G}_{-\tilde{\alpha}^1}^- \cdots \ddot{G}_{-\tilde{\alpha}^l}^-) \sum_{w \in W} q^{-(b, \rho_k + w(\rho_k))} \Gamma_{w^{-1}(b)} \zeta_w.$$

We claim that for any $b \in P_+$,

$$(5.25) \quad \xi^\delta(Y_{-b}) \stackrel{\text{def}}{=} \mathfrak{ae}^\delta(f_{-\tilde{\alpha}^1} \cdots f_{-\tilde{\alpha}^l}) \sum_{w \in W} q^{-(b, \rho_k + w(\rho_k))} \Gamma_{w^{-1}(b)} \zeta_w.$$

is regular at $t_\nu = 0$. The proof is similar to that for $\xi^\delta(Y_b)$ from (5.14) that was given before step (1). One uses (5.22) instead of (5.15).

(3) In the case of $b = \omega_r$ ($r \in O'$), the regularity of $\mathfrak{ae}^\delta(Y_{-b})$ is immediate from that of $\xi^\delta(Y_{-b})$, due to Proposition 5.1(ii).

(4) For b equal to any negative short root α , the proof is similar to (2). Use Lemma 1.4 to choose a reduced decomposition $s_\vartheta = s_{j_1} \cdots s_{j_p} s_m s_{j_p} \cdots s_{j_1}$ such that $s_{j_r} \cdots s_{j_1}(-\vartheta) = -\alpha$ where $0 \leq r \leq p$. Then, starting from $Y_\vartheta^{-1} = T_{s_\vartheta}^{-1} T_0^{-1}$, we use (1.33) and (1.45) to get

$$\begin{aligned} Y_{s_{j_1}(\vartheta)}^{-1} &= T_{j_1} Y_\vartheta^{-1} T_{j_1}, \quad Y_{s_{j_2} s_{j_1}(\vartheta)}^{-1} = T_{j_2} T_{j_1} Y_\vartheta^{-1} T_{j_1} T_{j_2}, \quad \dots, \\ Y_\alpha^{-1} &= T_{j_r} \cdots T_{j_1} Y_\vartheta^{-1} T_{j_1} \cdots T_{j_r}. \end{aligned}$$

Then, as in (2), the regularity of $\mathfrak{ae}^\delta(Y_\alpha^{-1})$ can be shown using Proposition 5.1(ii, iv). \square

5.4. Other generators. Recall the definition of $\mathfrak{H}^{b, \varphi}$ from Section 3.6 (we continue to take $B = P$). We will calculate the RE^δ limits of the remaining generators \check{T}_i ($i \geq 0$) and $\check{\Pi}$. Recall that $\check{T}_i = \check{T}_i$ for $i > 0$ and $\check{T}_0 = \varphi(\check{T}_0) = t_0^{1/2} T_{s_\vartheta}^{-1} T_0^{-1}$. We also consider $\tilde{X}_b \stackrel{\text{def}}{=} q^{(b_+, \rho_k)} X_b = \varphi(\check{Y}_{-b})$.

Proposition 5.5. (i) The operators $\hat{T}_i = RE^\delta(\check{T}_i)$ exist for all $i = 0, \dots, n$. Moreover,

$$(5.26) \quad \hat{T}_i = RE^\delta(\check{T}_i) = \sum_{\substack{w \in W \text{ s.t.} \\ w^{-1}(\alpha_i) < 0}} \zeta_w (s_i - 1) \quad \text{for } i > 0.$$

(ii) For any $b \in P$,

$$(5.27) \quad RE^\delta(\tilde{X}_b) = \sum_{\substack{w \in W \text{ s.t.} \\ w^{-1}(b) = b_+}} X_{b_+} \zeta_w.$$

(iii) For any $r \in O'$,

$$(5.28) \quad \begin{aligned} RE^\delta(\pi_r^{-1}) &= \sum_{w \in W} \left(X_{w^{-1}(\omega_r)} \prod_{\substack{\alpha \in \lambda(u_r) \text{ s.t.} \\ (w^{-1}(\alpha), \rho^\vee) = 1}} (1 - X_{w^{-1}(\alpha)}^{-1}) \zeta_w \right) u_r^{-1} \\ &+ \sum_{v < u_r^{-1}} \left(\sum_{w \in W} f_{v,w} \zeta_w \right) v \text{ for certain } f_{v,w} \in \mathbb{Q}'_q[X_b, b \in B]. \end{aligned}$$

Proof. (i) Using $\mathfrak{ae}^\delta(s_i) = s_i$ and (3.35), we readily arrive at (5.26) for $i > 0$. The case of $i = 0$ is significantly more involved. We have $\check{T}_0 = \varphi(\check{T}_0) = t_0^{1/2} T_{s_\vartheta}^{-1} X_\vartheta^{-1}$, where φ is the duality anti-involution defined in (1.43). Write $s_\vartheta = s_{j_l} \cdots s_{j_1} = s_{j_1} \cdots s_{j_l}$ ($l = l(s_\vartheta)$). Let $\alpha^p = s_{j_1} \cdots s_{j_{p-1}}(\alpha_{j_p}) \in \lambda(s_\vartheta)$ for $p = 1, \dots, l$.

Now

$$(5.29) \quad \check{T}_0 = t_0^{1/2} \prod_{\nu} t_{\nu}^{-l_{\nu}(s_\vartheta)/2} \check{G}_{-\alpha^1}^{-1} \cdots \check{G}_{-\alpha^l}^{-1} s_\vartheta X_\vartheta^{-1}.$$

By Lemma 1.3(i), one has $l_{\nu}(s_\vartheta) = 2(\vartheta, \rho_{\nu}^\vee) - \delta_{\nu, \vartheta}$. Hence

$$t_0^{1/2} \prod_{\nu} t_{\nu}^{-l_{\nu}(s_\vartheta)/2} = \prod_{\nu} t_{\nu}^{-(\vartheta, \rho_{\nu}^\vee) + \delta_{\nu, \vartheta}}.$$

Returning to (5.29), we have

$$(5.30) \quad \mathfrak{ae}^\delta(\check{T}_0) = \mathfrak{ae}^\delta(\check{G}_{-\alpha^1}^{-1} \cdots \check{G}_{-\alpha^l}^{-1}) \sum_{w \in W} t_{\text{sht}} q^{-(\vartheta, \rho_k + w(\rho_k))} X_{w^{-1}(\vartheta)} \zeta_w s_\vartheta.$$

By Proposition 5.1(iv),

$$\begin{aligned} &{}^\dagger \text{ord}_w^\nu(\mathfrak{ae}^\delta(\check{G}_{-\alpha^1}^{-1} \cdots \check{G}_{-\alpha^l}^{-1})) \\ &\geq \text{ord}_w^\nu(\mathfrak{ae}^\delta(f_{-\alpha^1} \cdots f_{-\alpha^l})) = \sum_{\alpha \in \lambda(s_\vartheta) \cap (R_+ \setminus \lambda(w^{-1}))} \delta_{\alpha, \nu}. \end{aligned}$$

The claim now follows from (5.22) and the description of the sets $\lambda_\nu(\vartheta)$ due to $\vartheta = s_0 s_\vartheta$. The t_{sht} factor in (5.30) accounts for the case when $w^{-1}(\vartheta) > 0$, because $\lambda(s_\vartheta) = \lambda(\vartheta) \setminus \{[\vartheta, 1]\}$.

(ii) By definition, $\tilde{X}_b = q^{(b_+, \rho_k)} X_b$; hence

$$\mathfrak{ae}^\delta(\tilde{X}_b) = \sum_{w \in W} q^{(b_+ - w^{-1}(b), \rho_k)} X_{w^{-1}(b)} \zeta_w.$$

Now (5.27) follows, using the fact that $b_+ \geq w^{-1}(b)$ for all $w \in W$.

(iii) Recall that $\check{\pi}_r^{-1} = X_{\omega_r} T_{u_r^{-1}}$. Let $u_r = s_{j_l} \cdots s_{j_1}$ be a reduced decomposition. Construct $\lambda(u_r) = \{\alpha^1, \dots, \alpha^l\}$ using this decomposition. Then

$$\check{\pi}_r^{-1} = q^{-(\omega_r, \rho_k)} X_{\omega_r} \ddot{G}_{-\alpha^1}^+ \cdots \ddot{G}_{-\alpha^l}^+ u_r^{-1}.$$

We have used here that $l_\nu(u_r) = l_\nu(\omega_r) = 2(\omega_r, \rho_\nu^\vee)$. Hence

$$\mathfrak{ae}^\delta(\check{\pi}_r^{-1}) = \left(\sum_{w \in W} q^{-(\omega_r, \rho_k + w(\rho_k))} X_{w^{-1}(\omega_r)} \zeta_w \right) \mathfrak{ae}^\delta(\ddot{G}_{-\alpha^1}^+ \cdots \ddot{G}_{-\alpha^l}^+) u_r^{-1}.$$

Now, by (5.22) and Proposition 5.1(ii), the limit $RE^\delta(\check{\pi}_r^{-1})$ exists. Then (5.28) follows readily. \square

Propositions 5.4 and 5.5 provide a direct justification, independent of Theorem 3.4, of one of the key results of this paper: the action of $\overline{\mathcal{H}}^{b, \varphi}$, the limit of $\check{\mathcal{H}}^{b, \varphi}$ as $t_\nu \rightarrow 0$, in $\text{Spin}(\mathbb{Q}'_q(X))$. Moreover, we obtain that the generators of $\overline{\mathcal{H}}^{b, \varphi}$ have no nontrivial denominators and therefore preserve $\text{Spin}(\overline{\mathcal{V}})$, as in Part (i) of Theorem 3.4. Finally, we also see that it is not necessary to work over the field $\mathbb{Q}'_q = \mathbb{Q}(q^{1/(2m)})$; the action of the generators of $\overline{\mathcal{H}}^{b, \varphi}$, including that of the Toda-Dunkl operators, is defined over the ring $\mathbb{Z}[q^{\pm 1/(2m)}]$.

Symmetrization. Let $\text{Spin}^\delta(\overline{\mathcal{V}})$ denote the space of W -invariants of $\text{Spin}(\overline{\mathcal{V}})$ under the δ -action, which is simply $\delta(\overline{\mathcal{V}})$. Recall that RE (without the super index δ) is the non-spinor Ruijsenaars-Etingof procedure defined in (3.1). We will use the operators \mathcal{L}_f and L_f from (2.20), where $f \in \mathbb{Q}'_q[X]^W$.

Proposition 5.6. *Upon the restriction to $\text{Spin}^\delta(\overline{\mathcal{V}})$, one has*

$$RE^\delta(\mathcal{L}_f) = \sum_{w \in W} RE(L_f) \zeta_w,$$

for any $f \in \mathbb{Q}'_q[X]^W$.

Proof. First, $f(Y)$ is central in $\check{\mathcal{H}}_Y^\varphi$, the subalgebra of $\check{\mathcal{H}}^\varphi$ generated by $\check{T}_i = T_i$ ($i > 0$) and Y_b ($b \in P$), and hence $RE^\delta(\mathcal{L}_f)$ commutes with $RE^\delta(\check{T}_i)$ from (5.26) for $i = 1, \dots, n$. Second, an element

$g \in \text{Spin}(\overline{\mathcal{V}})$ belongs to $\text{Spin}^\delta(\overline{\mathcal{V}})$ if and only if $\widehat{T}_i(g) = 0$ for $i = 1, \dots, n$. Indeed, applying (5.26) to $g = \sum_{w \in W} g_w \zeta_w$ gives

$$\widehat{T}_i(g) = \sum_{\substack{w \in W \text{ s.t.} \\ w^{-1}(\alpha_i) < 0}} (g_{s_i w} - g_w) \zeta_w.$$

The right-hand side vanishes if and only if $g_{s_i w} = g_w$ whenever one has $w^{-1}(\alpha_i) < 0$. The latter condition is always met either by w or by $w' = s_i w$. Thus all differences $g_{s_i w} - g_w$ must vanish.

We conclude that $RE^\delta(\mathcal{L}_f)$ preserves $\text{Spin}^\delta(\mathcal{V})$. Therefore it has the form $\sum_{w \in W} M \zeta_w$ upon the restriction to $\text{Spin}^\delta(\overline{\mathcal{V}})$ for some difference operator M . By considering the id-component of $RE^\delta(\mathcal{L}_f)$, one sees that $M = RE(L_f)$. \square

5.5. Examples. *Toda-Dunkl operators.* For the root system A_1 ,

$$\begin{aligned} \widehat{Y}_\omega &= \Gamma_{-\omega}^e \left((\zeta_{\text{id}} + (1 - X_\alpha^{-1}) \zeta_s) \text{id} + (-\zeta_{\text{id}} + X_\alpha^{-1} \zeta_s) s \right), \\ \widehat{Y}_\omega^{-1} &= \widehat{Y}_{-\omega} = \left((1 - X_\alpha^{-1}) \zeta_{\text{id}} + \zeta_s \right) \Gamma_\omega^e + \left(\zeta_{\text{id}} - X_\alpha^{-1} \zeta_s \right) \Gamma_{-\omega}^e s, \end{aligned}$$

in terms of the fundamental weight ω and simple root α , where $s = s_\alpha$ and $\Gamma_{-\omega}^e$ is from (3.30). Upon the restriction to $\text{Spin}^\delta(\overline{\mathcal{V}})$, one has

$$\widehat{Y}_\omega + \widehat{Y}_\omega^{-1} = \left((1 - X_\alpha^{-1}) \Gamma_\omega + \Gamma_{-\omega} \right),$$

a special case of Proposition 5.6.

For the root system A_2 , one has

$$\begin{aligned} \widehat{Y}_{\omega_1} &= \Gamma_{-\omega_1}^e \left((\zeta_{\text{id}} + (1 - X_{\alpha_1}^{-1}) \zeta_{s_1} + \zeta_{s_2} + (1 - X_{\alpha_2}^{-1}) \zeta_{s_1 s_2} \right. \\ &\quad \left. + (1 - X_{\alpha_1}^{-1}) \zeta_{s_2 s_1} + (1 - X_{\alpha_2}^{-1}) \zeta_{s_1 s_2 s_1}) \text{id} \right. \\ &\quad \left. + (-\zeta_{\text{id}} + X_{\alpha_1}^{-1} \zeta_{s_1} - \zeta_{s_2} - (1 - X_{\alpha_1}^{-1}) \zeta_{s_2 s_1} + X_{\alpha_2}^{-1} \zeta_{s_1 s_2 s_1}) s_1 \right. \\ &\quad \left. + (\zeta_{s_2} + X_{\alpha_1 + \alpha_2}^{-1} \zeta_{s_1 s_2} - X_{\alpha_1}^{-1} \zeta_{s_2 s_1} - X_{\alpha_1 + \alpha_2}^{-1} \zeta_{s_1 s_2 s_1}) s_1 s_2 \right. \\ &\quad \left. + (-\zeta_{s_1} - \zeta_{s_2} + (1 - X_{\alpha_1}^{-1}) X_{\alpha_2}^{-1} \zeta_{s_1 s_2} + X_{\alpha_1}^{-1} \zeta_{s_2 s_1} + X_{\alpha_1 + \alpha_2}^{-1} \zeta_{s_1 s_2 s_1}) s_1 s_2 s_1 \right). \end{aligned}$$

The operator \widehat{Y}_{ω_2} is obtained by interchanging the indices 1 and 2 of ω_i , s_i , and α_i in the above formula. The operators \widehat{Y}_{ω_i} are invertible (their inverses are $\widehat{Y}_{-\omega_i}$), and one has

$$\widehat{Y}_{\omega_1}^{-1} + \widehat{Y}_{\omega_1} \widehat{Y}_{\omega_2}^{-1} + \widehat{Y}_{\omega_2} = (1 - X_{\alpha_1}^{-1}) \Gamma_{\omega_1} + (1 - X_{\alpha_2}^{-1}) \Gamma_{-\omega_1 + \omega_2} + \Gamma_{-\omega_2}$$

upon the restriction to $\text{Spin}^\delta(\overline{\mathcal{V}})$, where the right-hand side is the q -Toda operator $RE(L_{-\omega_1})$; cf. Proposition 5.6 and (2.22).

For the root system B_2 , with α_1 long and α_2 short, the fundamental weight ω_2 is minuscule, while $\omega_1 = \vartheta$ is not. One has

$$\begin{aligned} \widehat{Y}_{\omega_2} = & \Gamma_{-\omega_2}^{\varrho} \left((\zeta_{\text{id}} + \zeta_{s_1} + (1 - X_{\alpha_1}^{-1})(\zeta_{s_2 s_1} + \zeta_{s_1 s_2 s_1})) \right. \\ & \left. + (1 - X_{\alpha_2}^{-1})(\zeta_{s_2} + \zeta_{s_1 s_2} + \zeta_{s_2 s_1 s_2} + \zeta_{s_1 s_2 s_1 s_2}) \right) \text{id} \\ & + (-(\zeta_{\text{id}} + \zeta_{s_1}) + X_{\alpha_2}^{-1}(\zeta_{s_2} + \zeta_{s_1 s_2 s_1 s_2}) - (1 - X_{\alpha_2}^{-1})\zeta_{s_1 s_2} \\ & \quad - (1 - X_{\alpha_1}^{-1})\zeta_{s_1 s_2 s_1}) s_2 \\ & + (\zeta_{s_1} + (1 - X_{\alpha_2}^{-1})\zeta_{s_1 s_2} + X_{\alpha_1 + \alpha_2}^{-1}\zeta_{s_2 s_1} - X_{\alpha_1}^{-1}\zeta_{s_1 s_2 s_1} \\ & \quad + X_{\alpha_1 + \alpha_2}^{-1}(1 - X_{\alpha_2}^{-1})\zeta_{s_2 s_1 s_2} - X_{\alpha_1 + \alpha_2}^{-1}\zeta_{s_1 s_2 s_1 s_2}) s_2 s_1 \\ & + (-(\zeta_{s_1} + \zeta_{s_2}) - (1 - X_{\alpha_2}^{-1})\zeta_{s_1 s_2} + X_{\alpha_1}^{-1}(1 - X_{\alpha_2}^{-1})\zeta_{s_2 s_1} \\ & \quad + X_{\alpha_1 + 2\alpha_2}^{-1}\zeta_{s_2 s_1 s_2} + X_{\alpha_1}^{-1}\zeta_{s_1 s_2 s_1}) s_2 s_1 s_2 \\ & + (-(\zeta_{s_1} + \zeta_{s_2 s_1}) + X_{\alpha_2}^{-1}\zeta_{s_1 s_2} + X_{\alpha_2}^{-1}(1 - X_{\alpha_1}^{-1})\zeta_{s_2 s_1 s_2} \\ & \quad + X_{\alpha_1 + \alpha_2}^{-1}\zeta_{s_1 s_2 s_1} - X_{\alpha_1 + \alpha_2}^{-1}(1 - X_{\alpha_2}^{-1})\zeta_{s_1 s_2 s_1 s_2}) s_1 s_2 s_1 \\ & \left. + (\zeta_{s_1} - X_{\alpha_2}^{-1}\zeta_{s_1 s_2} - X_{\alpha_1 + \alpha_2}^{-1}\zeta_{s_1 s_2 s_1} + X_{\alpha_1 + 2\alpha_2}^{-1}\zeta_{s_1 s_2 s_1 s_2}) s_1 s_2 s_1 s_2 \right). \end{aligned}$$

Application to the nonsymmetric Whittaker function. Let us give the values of the coefficients $a_{b,w}$ from (2.60) and Proposition 3.3 in the case of the root system A_2 . These coefficients are the only ingredient of the theory of \overline{E}^\dagger -polynomials necessary for an explicit description of the nonsymmetric Whittaker function Ω . See [CM] or [CO2, (2.7)] for the A_1 -case.

Note that $a_{0,w} = 1$ for all $w \in W$. The following tables give the values of $a_{b,w}$ for nonzero b with a fixed b_- and all 6 elements $w \in \mathbf{S}_3$.

For $b_- = n\omega_2$ ($n < 0$):

| $b \setminus w$ | id | s_1 | s_2 | $s_2 s_1$ | $s_1 s_2$ | $s_1 s_2 s_1$ |
|------------------|----|-------|-------|-----------|-----------|---------------|
| $b = n\omega_2$ | 1 | 1 | q^n | q^n | q^n | q^n |
| $b = s_2(b_-)$ | 0 | 0 | 1 | 1 | q^n | q^n |
| $b = -n\omega_1$ | 0 | 0 | 0 | 0 | 1 | 1 |

We put $a_{b,w}$ in the corresponding row and column above and in the following tables.

For $b_- = n\omega_1$ ($n < 0$):

| $b \setminus w$ | id | s_1 | s_2 | s_2s_1 | s_1s_2 | $s_1s_2s_1$ |
|------------------|----|-------|-------|----------|----------|-------------|
| $b = n\omega_1$ | 1 | q^n | 1 | q^n | q^n | q^n |
| $b = s_1(b_-)$ | 0 | 1 | 0 | q^n | 1 | q^n |
| $b = -n\omega_2$ | 0 | 0 | 0 | 1 | 0 | 1 |

Note that one can pass from either of these two tables to the other by relabeling the indices 1 and 2 in the columns.

For $b_- = n_1\omega_1 + n_2\omega_2$ ($n_1, n_2 < 0$):

| $b \setminus w$ | id | s_1 | s_2 | s_2s_1 | s_1s_2 | $s_1s_2s_1$ |
|-----------------|----|-----------|-----------|---------------|---------------|---------------|
| b_- | 1 | q^{n_1} | q^{n_2} | $q^{n_1+n_2}$ | $q^{n_1+n_2}$ | $q^{n_1+n_2}$ |
| $s_1(b_-)$ | 0 | 1 | 0 | q^{n_1} | $q^{n_1+n_2}$ | 0 |
| $s_2(b_-)$ | 0 | 0 | 1 | $q^{n_1+n_2}$ | q^{n_2} | 0 |
| $s_2s_1(b_-)$ | 0 | 0 | 0 | 1 | 0 | q^{n_2} |
| $s_1s_2(b_-)$ | 0 | 0 | 0 | 0 | 1 | q^{n_1} |
| b_+ | 0 | 0 | 0 | 0 | 0 | 1 |

Furthermore, let us provide the values of $a_{b,w} = q^{n_b(w(b))}$ for A_3 in the notation from Conjecture 2.7 for $b = b_-$, i.e. for antidominant b . Confirming this conjecture, $-n_b(w(b))$ coincides with the lowest q -degree of the coefficient of $X_{w(b)-b}$ in the product $\prod_{\alpha \in R_+} (1 - qX_\alpha)^{-1}$ from Lusztig's definition of the (nonaffine) Kostant q -partition function. See [JLZ] and [FFL] concerning using the Kostant q -partition function in the theory of the BK-filtration and the PBW-filtration.

Namely, $n_b(w(b)) = (b, \gamma_w)$ for $w(b) \succ b \in P_-$ and proper $\gamma_w \in P_+$. In this case, γ_w is the maximal positive root in the set $\lambda(w)$ ($\gamma_w = 0$ for $w = \text{id}$), except for the following permutations:

$$(5.31) \quad \begin{aligned} &\gamma_w = \epsilon_{12} + \epsilon_{34} \quad \text{for } w = (2143), \quad \gamma_w = \epsilon_{14} \quad \text{for } w = (3142), \\ &\gamma_w = \epsilon_{14} + \epsilon_{23} \quad \text{for } w = (3412), (4312), (3421), (4321), \end{aligned}$$

where $\epsilon_{ij} = \epsilon_i - \epsilon_j$, $\alpha_i = \epsilon_i - \epsilon_{i+1}$ in the notation from the tables of [B].

We note that one can compute the \overline{E}^\dagger -polynomials for A_n using the SAGE software for the E -polynomials based on the formula due to Haiman-Haglund-Loehr followed by $t \rightarrow \infty$. However a direct usage of the intertwining operators of nil-DAHA is more efficient (and we need them for all root systems).

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